Noncommutative Spaces and Schemes

Introduction

1. Spaces and algebras. The correspondence between spaces and commutative algebras is a fundamental fact of functional analysis discovered by I. M. Gelfand in the late thirties. A. Connes extended this correspondence to the noncommutative setting identifying 'noncommutative spaces' with noncommutative C^*-algebras. This lead to the creation of noncommutative differential geometry [C1], [C2].

The category of affine spaces in noncommutative algebraic geometry was defined by V. Drinfeld [Dr] (who calls them 'quantum spaces') in a similar way, as the category dual to the category of algebras, forcing to the noncommutative case the duality [algebras $\leftrightarrow$ affine spaces] of commutative algebraic geometry.

2. Spaces and categories. Noncommutative projective spaces were introduced by Yu.I. Manin [M1] via a formal extension of the Serre's description of the category of quasi-coherent sheaves on a projective space [S]: the category of quasi-coherent sheaves on the projective spectrum of an associative graded algebra R is the quotient category of the category of graded R-modules by the subcategory of locally finite ones (this approach was futher developed in [V1], [V2], [A2], [AZ], [OW], and in a number of other works). Thus a noncommutative projective space X over k is represented by a category, $Qcoh_X$, regarded as the category of quasi-coherent sheaves on X, together with a (global section) functor $Qcoh_X \longrightarrow k-mod$.

3. Affine schemes over a category. This viewpoint is well adapted to the affine case: for any associative ring R, the category $Qcoh_{\mathbf{Spec}(R)}$ is identified with the category $R-mod$ of left R-modules, and to any morphism $\mathbf{Spec}(R) \longrightarrow \mathbf{Spec}(S)$, there corresponds a (direct image) functor $Qcoh_{\mathbf{Spec}(S)} \longrightarrow Qcoh_{\mathbf{Spec}(R)}$. Here we identify the category $\mathbf{Aff}$ of affine noncommutative schemes with the category $\mathbf{Rings}^{op}$ dual to the category of rings, and denote by $\mathbf{Spec}(R)$ the object of $\mathbf{Aff}$ corresponding to the ring R.

One can characterise direct image functors of affine morphisms in abstract nonsense terms obtaining this way a notion of an *affine scheme* over an arbitrary category C (cf. [R3], or Section 2 of this work). If C is the category of left modules over a ring R, we recover affine noncommutative schemes over $\mathbf{Spec}(R)$.

4. Noncommutative algebras and non-symmetric monoidal categories. Local objects of commutative algebraic geometry over $\mathbf{Spec}(k)$ are commutative algebras in the symmetric monoidal category $k-mod^{\sim} = (k-mod, \otimes_k, k)$ of left k-modules. Local objects of the super-geometry are commutative algebras in the symmetric monoidal category of $\mathbb{Z}/2\mathbb{Z}$-graded k-modules. Affine noncommutative schemes over $\mathbf{Spec}(R)$ are defined by ring morphisms $R \longrightarrow S$. These are naturally identified with algebras in the monoidal category $R-bimod^{\sim} = (R-bimod, \otimes_R, R)$ of R-bimodules. Note that if the

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ring R is noncommutative, the category $R - bimod^{\sim}$ is not symmetric (or braided) any more, so that the notion of a commutative algebra in $R - bimod^{\sim}$ is not defined.

5. Affine schemes in a monoidal category. Quasi-coherent modules. For an arbitrary monoidal category $\mathcal{A}^{\sim}$, we define the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$ as the category dual to the category of algebras in $\mathcal{A}^{\sim}$. A link between the formal notion of an affine scheme in a monoidal category and a more geometric notion of an affine scheme over a category C is given by fixing an action, $\mathfrak{C} = (\mathcal{A}^{\sim}, C, \Phi : \mathcal{A} \longrightarrow EndC)$, of the monoidal category on the category C. This action allows to define for any algebra S in the monoidal category $\mathcal{A}^{\sim}$, the category $S - mod_C$ *of S-modules in C* which is called the category of *quasi-coherent modules on* $\mathbf{Spec}(S)/\mathfrak{C}$, and the forgetful functor $S - mod_C \longrightarrow C$ regarded as a direct image functor of the morphism of $\mathbf{Spec}(S)$ to the base.

A standard noncommutative example is the monoidal category $R - bimod^{\sim} = (R - bimod, \otimes_R, R)$ of bimodules over an associative ring R acting (by tensoring over R) on the category $R - mod$ of left R-modules. For any algebra S in the monoidal category $R - bimod^{\sim}$ (that is a ring morphism $R \longrightarrow S$), the category of quasi-coherent modules on $\mathbf{Spec}(S)/\mathfrak{C}$ is naturally equivalent to the category $S - mod$ of left S-modules.

A non-trivial example is given by the monoidal category $\mathcal{A}^{\sim} = \mathbf{S} - Vec_k^{\sim}$ of $\mathbf{S}$-*spaces* objects of which are families of representations of all symmetric groups, $\mathbf{S}_n$, $n \geq 1$, in vector spaces over a field k, and the tensor product is the so called *plethysm product.* Algebras in the monoidal category $\mathbf{S} - Vec_k^{\sim}$ are called *k-linear operads.* The monoidal category of $\mathbf{S}$-spaces acts canonically on the category $C = Vec_k$ of k-vector spaces. For each k-linear operad $\mathcal{P}$, the corresponding category of quasi-coherent sheaves on $\mathbf{Spec}(\mathcal{P})$ is called *the category of $\mathcal{P}$-algebras.*

6. Noncommutative spaces and schemes. Commutative schemes, or more general "varieties" (like algebraic spaces of Artin and Moishezon) are "spaces" obtained by glueing affine schemes. The data required to give a sense to such definition consists of (a) a category **Spaces**, (b) a functor from the category $\mathbf{Aff} = \mathbf{Rings}^{op}$ to the category **Spaces**, (c) a topology on **Aff**. Two categories of spaces are used: the category of locally ringed spaces and the category of sheaves of sets on the site $(\mathbf{Aff}, \mathfrak{T}_{fl})$, where $\mathfrak{T}_{fl}$ is a flat topology. Schemes are spaces (in any of the two senses) obtained by glueing affine schemes for the Zariski topology. The algebraic spaces of Artin and Moishenson are sheaves of sets obtained by glueing affine schemes for the étale topology.

The known examples of noncommutative spaces appear either as categories (of quasi-coherent sheaves) over a base category, like the **Proj** of a graded noncommutative ring mentioned in 2 above, or the flag variety of a quantized enveloping algebra (see [R3], [LR2]), or as sheaves of sets on the category of noncommutative affine schemes, like projective spaces and Grassmannians introduced in [KR]. Accordingly, we introduce the notions of a scheme and that of a locally affine space (noncommutative projective spaces of [KR] are locally affine, but not schemes or algebraic spaces) for appropriate choices of the category **Spaces** and a topology on the category of noncommutative affine schemes.

7. Virtually noncommutative geometries. The main purpose of this paper is to introduce noncommutative affine schemes in an appropriate way, define general 'locally affine spaces', in particular schemes, as 'spaces' glued from affine schemes, and give some important examples of noncommutative schemes. This occupies the most of the first six sections of the paper. The geometry developed in these sections is really noncommutative: commutative schemes enter into the picture, but the basic categorical operations, like taking direct products, send commutative schemes into noncommutative ones. For instance, the product of two copies of a line, $Spec\mathbb{C}[x]$, is the noncommutative affine scheme corresponding to the free algebra $\mathbb{C}\langle x, y\rangle$. We recover back the commutative geometry, actually a whole family of commutative geometries depending on the choice of a symmetry, β, of the base monoidal category (see above), using the 'abelianization functor'. This functor assigns to a noncommutative scheme X a β-commutative scheme, X_β, and a closed immersion $X_\beta \longrightarrow X$. In the case of the base $(k-mod, k-mod^\sim)$ and the standard symmetry $\beta : x \otimes y \longmapsto y \otimes x$, the abelianization functor maps an affine noncommutative scheme $\mathbf{Spec}(R)$ to the commutative scheme $\mathbf{Spec}(R_\beta)$, where R_β is the quotient of R by its commutant.

Taking infinitesimal neighborhoods of X_β in X defines 'nilpotezation functors' mapping noncommutative schemes to schemes which are 'virtually noncommutative', i.e. they are closed subschemes of formal noncommutative neighborhoods of commutative schemes. As a special case of this construction, we recover the 'non-commutative geometry based on commutator expansions' introduced recently by Kapranov [K].

The paper is organized as follows.

In Section 1 we recall 'geometric language' in which spaces are categories and morphisms are represented by (equivalence classes of) right exact functors called *inverse image functors*. This language appeared in [R], Ch.VII, and was used in [R3]. We introduce also the notion of a quasi-coherent presheaf.

In Section 2 we define affine morphisms and affine schemes over a category and reproduce analogs of main basic facts on conventional affine schemes in the new, noncommutative, context. In particular, we show that the category of affine schemes over a category C is dual to the category of so called *continuous* monads on the category C. If C is the category of R-modules, then the category of continuous monads on C is naturally equivalent to the category of rings over the ring R; in particular, if $R = \mathbb{Z}$, it is equivalent to the category of rings.

In Section 3 we define schemes in monoidal categories and the categories of quasi-coherent sheaves related to actions of monoidal categories.

In Sections 4 we develop an abstract version of a formalism of glueing new 'locally affine spaces' starting from what we call a *quasi-topology*. In Section 5 we apply the formalism of Section 4 to introduce schemes and locally affine spaces over a given base.

Section 6 is dedicated to noncommutative versions of well known and important schemes: vector fibers, the group schemes GL_V, Grassmannians, flag varieties. Similarly to their commutative prototypes, they are defined as contravariant functors on the category of affine schemes. We show that these functors are representable.

It is worth to mention that most of interesting and natural examples of noncommutative spaces are not schemes. For instance, the Grassmannian presented in this work is a subspace of the natural Grassmannian introduced in [KR]. The latter is a locally affine space, where 'locally' is understood in the sense of a flat topology.

In Section 7 we introduce *abelianization* and *nilpotezation* functors corresponding to a choice of a symmetry in the base monoidal category.

One of the purposes of this paper is to give a background to a series of works on noncommutative algebraic geometry, some of them have already appeared (like [LR1,2,3], [KR]), others are in progress. Some notions and constructions in this paper are given in a bigger generality than it is strictly necessary for examples presented in the text. This generality pays back in the continuations, where we study other examples and general properties of noncommutative spaces (which are not schemes), formal noncommutative geometry, differential calculus (including differential operators and D-modules), smooth non-commutative manifolds, and some applications of noncommutative geometry to mathematical physics and representation theory.

The paper was strongly influenced by conversations with I.M. Gelfand, A. Connes, and particularly with M. Kontsevich. Thus, the starting point of the work was an attempt to find a geometric meaning of quasi-determinants and (noncommutative) quasi-plückerian coordinates introduced by Gelfand and Retakh [GR]. This lead to the defining noncommutative Grassmannians and flag varieties. The project we are working on with M. Kontsevich dictated the degree of generality of the approach. Talking with A. Connes and attending his lectures have also influenced the setting of this work. I would like to thank D. Orlov for pointing out an error and for many useful discussions.

A considerable part of this work (actually its first version) was made during my staying at IHES. The present version was written at MPIM in Bonn. I would like to thank IHES and MPIM for the hospitality and excellent working conditions.

1. Categories as noncommutative 'spaces'.

Categories here are thought as categories of quasi-coherent or coherent sheaves on 'spaces' and are identified with the 'spaces'. The equivalence classes of some functors represent morphisms.

1.0. Preliminaries on fibered categories. Fix a category B. Recall that a *pseudo-functor* from B to Cat is the following data:

(a) A map $X \longmapsto \mathcal{F}_X$ from ObB to $ObCat$.

(b) A map which assigns to any morphism $f : X \longrightarrow Y$ of B, a functor $f^* : \mathcal{F}_X \longrightarrow \mathcal{F}_Y$.

(c) A map which assigns to any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, a functor morphism $c_{g,f} : f^*g^* \longrightarrow (gf)^*$.

This data satisfies the following conditions:

(i) $id_X^* = id_{\mathcal{F}_X}$.

(ii) $c_{f,g}$ is identical if f or g is identical.

(iii) Given three composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, the diagram

$$\begin{array}{ccc} f^*(hg)^* & \xrightarrow{c_{hg,f}} & (hgf)^* \\ {\scriptstyle f^*c_{h,g}}\uparrow & & \uparrow{\scriptstyle c_{h,gf}} \\ f^*g^*h^* & \xrightarrow{c_{f,g}h^*} & (gf)^*h^* \end{array}$$

is commutative.

To each pseudo-functor $B \longrightarrow Cat$, there corresponds a prefibered category $\mathcal{F}$ over B defined as follows. Objects of $\mathcal{F}$ are pairs (X, M), where $X \in ObB$, $M \in Ob\mathcal{F}_X$. Morphisms from (X, M) to (Y, L) are pairs (f, ξ), where f is a morphism from X to Y and ξ is a morhism $f^*(L) \longrightarrow M$. The composition of $(f, \xi) : (X, M) \longrightarrow (Y, L)$ and $(g, \nu) : (Y, L) \longrightarrow (Z, N)$ is defined by $(g, \nu) \circ (f, \xi) := (gf, \nu \circ g^*\xi \circ c_{f,g}(N) : (gf)^*(N) \longrightarrow M)$.

We have a canonical *fiber functor* $\pi : \mathcal{F} \longrightarrow B$, $(X, M) \longmapsto X$, $(f, \xi) \longmapsto f$. This together with morphisms $\{c_{g,f} : f^*g^* \longrightarrow (gf)^*\}$ defines a *catégorie clivée* which is, therefore, a prefibered category. The prefibered category $\pi : \mathcal{F} \longrightarrow B$ is fibered iff all morphisms $c_{f,g}$ are isomorphisms.

1.0.1. Fibered category of quasi-coherent morphisms. Fix a fibered category $\pi : \mathcal{F} \longrightarrow B$. Denote by $\mathfrak{Qcoh}\mathcal{F}$ the subcategory of $\mathcal{F}$ formed by all morphisms (f, ξ) such that ξ is an isomorphism. The composition of the embedding $\mathfrak{Qcoh}\mathcal{F} \longrightarrow \mathcal{F}$ and the projection π is a fibered category which will be called *the fibered category of quasi-coherent morphisms of* $\mathcal{F} \longrightarrow B$.

1.0.2. Presheaves and quasi-coherent presheaves. A *presheaf* in a fibered category $\mathcal{F} \xrightarrow{\pi} B$ (or an $\mathcal{F}$*-presheaf*) is any section of $\mathcal{F} \xrightarrow{\pi} B$, i.e. any functor $G : B \longrightarrow \mathcal{F}$ such that $\pi_{\mathcal{F}} \circ G = Id_B$. The category $Pre\mathcal{F}$ of $\mathcal{F}$-presheaves is the full subcategory of the category of functors $B \longrightarrow \mathcal{F}$ generated by $\mathcal{F}$-presheaves.

We call an $\mathcal{F}$-presheaf *quasi-coherent* if it takes values in the subcategory $\mathfrak{Qcoh}\mathcal{F}$. The full subcategory of $Pre\mathcal{F}$ generated by quasi-coherent presheaves will be denoted by $PQcoh\mathcal{F}$.

1.0.3. Lemma. *Suppose the base B has a final object, X. Then the category $PQcoh\mathcal{F}$ is naturally equivalent to the fiber $\mathcal{F}_X$ of $\mathcal{F}$ over X.*

Proof. For any $U \in ObB$, denote by i_U the unique morphism $U \longrightarrow X$. An equivalence $\mathcal{F}_X \longrightarrow Qcoh\mathcal{F}$ assigns to any object M of $\mathcal{F}_{T(X)}$ the presheaf $M^{\sim} : B \longrightarrow Mod\mathcal{F}$ which sends any object U of B to the pair $(U, i_U^*(M))$ and any morphism $g : U \longrightarrow V$ into $(g, c_{i_V,g})$. A quasi-inverse functor assigns to any quasi-coherent presheaf $M^{\sim}$ its value on X. ■

1.0.4. Base change. Fix a fibered category $\mathcal{F}$ with a base B. Any functor $T : \mathcal{T} \longrightarrow B$ defines the *induced* fibered category $\mathcal{F}^T$ with the base T, usually denoted by $\mathcal{F} \times_B \mathcal{T}$. The fiber of $\mathcal{F}^T$ over $U \in Ob\mathcal{T}$ is $\mathcal{F}_{T(U)}$, the inverse image of an arrow f is $(Tf)^*$ and the cocycle $(gf)^* \longrightarrow f^*g^*$ is $c_{Tf,Tg}$.

Let $G : B \longrightarrow S$ be a functor. For any $X \in ObS$, we have the category G/X and the natural functor $T : G/X \longrightarrow B$. We denote the fibered category $\mathcal{F} \times_B G/X$ by $\mathcal{F}^{G/X}$, or by $\mathcal{F}^X$. The category of (resp. quasi-coherent) presheaves in $\mathcal{F}^X$ will be denoted by $Pre_X\mathcal{F}$ (resp. $PQoh_X\mathcal{F}$). Its objects will be called *(quasi-coherent) presheaves on X*. Since the construction is functorial in X, it gives a rise to a fibered category over S having $PQcoh_X\mathcal{F}$ as a fiber at an object X.

1.0.5. Continuous morphisms. Fix a fibered category $\mathcal{F}$ with a base B. We call a morphism $f : X \longrightarrow Y$ of the base *continuous* if it has a direct image functor. (Dually, we call a morphism of a base of a cofibered category *continuous* if it has an inverse image functor.) The composition of continuous morphisms is continuous, and all isomorphisms of the category B are continuous. Thus continuous morphisms define a subcategory B_c of the category B, hence a bifibered subcategory $\pi_c : \mathcal{F}_c \longrightarrow B_c$.

1.0.6. Monads and modules over monads. Recall that a monad in a category C is a pair (F, μ), where F is a functor $C \longrightarrow C$ and μ is a functor morphism $F^2 \longrightarrow F$ having the following properties: $\mu \circ F\mu = \mu \circ \mu F$ and there exists a morphism $\eta : Id_C \longrightarrow F$ such that $\mu \circ F\eta = id_F = \mu \circ \eta F$. The morphism η is uniquely defined and is called *the unit of the monad* (F, μ). Let $\mathbb{F} = (F, \mu)$ and $\mathbb{G} = (G, \nu)$ be monads in C with units resp. $\eta : Id_C \longrightarrow F$ and $\delta : Id_C \longrightarrow G$. A morphism from $\mathbb{F}$ to $\mathbb{G}$ is a functor moprhism $\phi : F \longrightarrow G$ such that $\phi \circ \eta = \delta$ and $\nu \circ \phi \odot \phi = \phi \circ \mu$. Here $\phi \odot \phi := G\phi \circ \phi F : F^2 \longrightarrow G^2$. The composition of morphisms is defined in an obvious way. We denote the category of monads in C by $\mathfrak{Mon}C$. The category $\mathfrak{Mon}C$ has a canonical initial object – the monad $\mathbb{I}_C = (Id_C, id)$.

Fix a monad $\mathbb{F} = (F, \mu)$ in C with the unit η. An $\mathbb{F}$-*module* is a pair (M, m), where $M \in ObC$ and m is a morphism $F(M) \longrightarrow M$ such that $m \circ \eta(M) = id_M$ and $m \circ Fm = m \circ \mu(M)$. A module morphism from (M, m) to (M', m') is a morphism $f : M \longrightarrow M'$ such that $m' \circ Ff = f \circ m$. The composition is induced by the composition in C. The category of $\mathbb{F}$-modules will be defined by $\mathbb{F} - mod$.

Thus modules over monads on a given category C provide a natural example of a cofibered category $\mathfrak{Mod}C$: its base is the category of monads $\mathfrak{Mon}C$ on C and the fiber at a monad $\mathbb{G}$ is the category $\mathbb{G} - mod$ of $\mathbb{G}$-modules. Note that $\mathfrak{Mod}C$ is not a fibered category in general, i.e. the pull-back functor corresponding to a monad morphism does not necessarily have a left adjoint.

Recall that a pair of arrows $g_1, g_2 : X \rightrightarrows Y$ is called *reflexive* if there exists a morphism $h : Y \longrightarrow X$ such that $g_1 \circ h = id_Y = g_2 \circ h$.

1.0.6.1. Lemma. *Suppose the category C has cokernels of reflexive pairs. Let $\mathbb{F} = (F, \mu)$ be a monad in C such that the functor F preserves cokernels of all reflexive pairs of arrows. Then for any monad morphism $\phi : \mathbb{G} \longrightarrow \mathbb{F}$ the corresponding 'pull-back' functor $\phi_\sharp : \mathbb{F} - mod \longrightarrow \mathbb{G} - mod$ is a direct image functor of a morphism $(\mathbb{G} - mod \longrightarrow C) \longrightarrow (\mathbb{F} - mod \longrightarrow C)$.*

Proof. For any $\mathbb{G}$-module $\mathcal{M} = (M, m)$, we set $\mathbb{F} \otimes_{\mathbb{G}} (\mathcal{M}) := cok(Fm, \mu \circ F\phi(M))$. Since F preserves cokernels of reflexive pairs of arrows, there is a unique $\mathbb{F}$-module structure m' on $M' = \mathbb{F} \otimes_{\mathbb{G}} (\mathcal{M})$ such that $(F(M), \mu) \longrightarrow (M', m')$ is an $\mathbb{F}$-module moprhism. ■

1.0.7. Continuous morphisms and monads. Fix a fibered category $\mathcal{F}$ with a base B. For any object S of the base B, denote by $(B/S)_c$ the full subcategory of the category B/S objects of which are pairs (X, f), where f is a continuous morphism $X \longrightarrow S$. Denote by $\mathcal{F}_c^S$ the fibered category with the base $(B/S)_c$ induced by the natural functor $(B/S)_c \longrightarrow B$. To each object (X, f) of the category $(B/S)_c$ there corresponds a monad $(\mathbb{G}_f, \mu_f) = (f_*f^*, f_*\epsilon_f f^*)$ on the category S.

1.0.7.1. Proposition. *The map $f \longmapsto (\mathbb{G}_f, \mu_f) := (f_*f^*, f_*\epsilon_f f^*)$ extends naturally to a morphism $\mathfrak{L}^* : \mathcal{F}_c^S \longrightarrow \mathfrak{Mod}\mathcal{F}_S$ of fibered categories.*

Proof. Let $h : A \longrightarrow B$ be a morphism from $f : \mathcal{A} \longrightarrow S$ to $g : \mathcal{B} \longrightarrow S$; i.e. $f = gh$. And let $c_{g,h} : h^*g^* \longrightarrow f^*$ be the corresponding isomorphism. The composition of the canonical morphisms

$$\eta_f g_* : g_* \longrightarrow f_*f^*g_*, \quad f_*c_{g,h}^{-1}g_* : f_*f^*g_* \longrightarrow f_*h^*g^*g_*, \quad \text{and} \quad f_*h^*\epsilon_g : f_*h^*g^*g_* \longrightarrow f_*h^*$$

provides a morphism

$$\phi_h = f_*c_{g,h} \circ (f_*h^*\epsilon_g \circ f_*c_{g,h}^{-1}g_* \circ \eta_f g_*)g^* : g_*g^* \longrightarrow f_*f^* \qquad (1)$$

We claim that ϕ_h is a monad morphism $\mathbb{G}_g \longrightarrow \mathbb{G}_f$, i.e. $\phi_h \circ \eta_g = \eta_f$ and $\mu_f \circ \phi_h \odot \phi_h = \phi_h \circ \mu_g$. In fact,

$$\phi_h \circ \eta_g = f_*c_{g,h} \circ (f_*h^*\epsilon_g \circ f_*c_{g,h}^{-1}g_* \circ \eta_f g_*)g^* \circ \eta_g = f_*c_{g,h} \circ f_*h^*\epsilon_g g^* \circ f_*c_{g,h}^{-1}g_*g^* \circ f_*f^*\eta_g \circ \eta_f =$$

$$f_*c_{g,h} \circ f_*h^*\epsilon_g g^* \circ f_*h^*g^*\eta_g \circ f_*c_{g,h}^{-1} \circ \eta_f = f_*c_{g,h} \circ f_*h^*(\epsilon_g g^* \circ g^*\eta_g) \circ f_*c_{g,h}^{-1} \circ \eta_f =$$

$$f_*c_{g,h} \circ f_*c_{g,h}^{-1} \circ \eta_f = \eta_f$$

since $\epsilon_g g^* \circ g^*\eta_g = id_{g^*}$.

We have

$$\phi_h \circ \mu_g = f_*c_{g,h} \circ (f_*h^*\epsilon_g \circ f_*c_{g,h}^{-1}g_* \circ \eta_f g_*)g^* \circ g_*\epsilon_g g^* =$$

$$f_*c_{g,h} \circ (f_*h^*\epsilon_g \circ f_*c_{g,h}^{-1}g_* \circ \eta_f g_* \circ g_*\epsilon_g)g^*$$

and

$$\mu_f \circ \phi_h \odot \phi_h =$$

$$f_*\epsilon_f f^* \circ f_* f^*(f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^*) \circ (f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^*) g_* g^* =$$

$$f_* c_{g,h} \circ f_* \epsilon_f h^* g^* \circ f_* f^*(f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^*) \circ (f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^*) g_* g^* =$$

$$f_*(c_{g,h} \circ h^* \epsilon_g \circ c^{-1}_{g,h} g_*) \circ f_* \epsilon_f f^* g_* \circ f_* f^* \eta_f g_*) g^*) \circ (f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^*) g_* g^* =$$

$$f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_*) g^* \circ (f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^*) g_* g^* =$$

$$f_* c_{g,h} \circ f_* h^* \epsilon_g g^* \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_*) g^* g_* g^* =$$

$$f_* c_{g,h} \circ (f_* h^* \epsilon_g \circ f_* c^{-1}_{g,h} g_* \circ \eta_f g_* \circ g_* \epsilon_g) g^*$$

In the last transformation we used the equality $\epsilon_g \circ \epsilon_g g^* g_* = \epsilon_g \circ g^* g_* \epsilon_g$.

The claimed morphism of fibered categories maps any object $(X \xrightarrow{f} S, M)$ of the category $Mod\mathcal{F}^S_c$ to the object $(\mathbb{G}_f, (f_*(M), \mu_f(M)))$ of the category $\mathfrak{Mod}\mathcal{F}_S$ and any morphism $(h, \xi) : (X \xrightarrow{f} S, M) \longrightarrow (Y \xrightarrow{g} S, N)$ of the category $Mod\mathcal{F}^S_c$ to the morphism $(\phi_h, \xi^\sim)$, where $\xi^\sim$ is a morphism canonically defined by ξ and $c^{-1}_{g,h}$. Details are left to the reader. ■

Let $Hom_2 B$ denote the class of all continuous morphisms $f : X \longrightarrow Y$ of the category B having the following properties:

(i) The category $\mathcal{F}_X$ has cokernels of all reflexive pairs of morphisms (recall that a pair of morphisms $g_1, g_2 : M \longrightarrow L$ is called *reflexive* if there exists a morphism $h : L \longrightarrow M$ such that $g_1 \circ h = id_L = g_2 \circ h$).

(ii) A direct image functor of f preserves cokernels of all reflexive pairs of morphisms and reflects isomorphisms.

One can see that $Hom_2 B$ is closed under composition and contains identical morphisms of all objects X with the property (i) above.

Denote by $(B/S)_2$ the full subcategory of the category B/S objects of which are pairs (X, f), where $f : X \longrightarrow S$ belongs $Hom_2 B$. Let $\mathcal{F}^S_2$ be the induced fibered subcategory of the fibered category $\mathcal{F}^S_c$.

1.0.7.2. Lemma. *The restriction $\mathfrak{L}_2$ of the morphism $\mathfrak{L}^*$ to the fibered subcategory $\mathcal{F}^S_2$ is fully faithful (i.e. it induces a fully faithful functor from the category $\mathcal{F}^S_2$ to the category of modules $\mathfrak{Mod}\mathcal{F}_S$).*

Proof. It follows from the Barr-Beck theorem (cf. [ML], VI.7) that the morphism $\mathfrak{L}_2$ induces an equivalence of categories on each fiber. The assertion follows now from SGA4, VI.4.2 (LNM 224, pp. 159–160). ■

1.0.8. Descent. Fix a fibered category $\mathcal{F}$ with a base B.

1.0.8.1. Comodules. Fix a set $\mathfrak{U} = \{U \xrightarrow{u} X\}$ of continuous morphisms of B. An $\mathfrak{U}$-*comodule* is a pair (F, ϕ), where F is a function which assigns to any $(U \xrightarrow{u} X) \in \mathfrak{U}$ an object F_u of $\mathcal{F}_U$ and ϕ assigns to any pair of arrows $U \xrightarrow{u} X \xleftarrow{v} V$ of $\mathfrak{U}$, a morphism $\phi_{u,v} : F_v \longrightarrow v^* u_*(F_u)$ such that the following conditions hold.

(a) $\epsilon_u \circ \phi_{u,u} = id_{F_u}$.

(b) For any three arrows of $\mathfrak{U}$, $u : U \longrightarrow X$, $v : V \longrightarrow X$, and $w : W \longrightarrow X$, the diagram

$$\begin{array}{ccc} F_w & \xrightarrow{\phi_{v,w}} & w^*v_*(F_v) \\ \phi_{u,w}\downarrow & & \downarrow w^*v_*\phi_{u,v} \\ w^*u_*(F_u) & \xrightarrow{w^*\eta_v u_*(F_u)} & w^*v_*v^*u_*(F_u) \end{array} \qquad (1)$$

is commutative.

For any two $\mathfrak{U}$-comodules (F, ϕ) and (G, ψ), a morphism from (F, ϕ) to (G, ψ) is a set of morphisms $\{g_u : F_u \longrightarrow G_u | u \in \mathfrak{U}\}$ such that the diagram

$$\begin{array}{ccc} F_u & \xrightarrow{g_u} & G_u \\ \phi_{v,u}\downarrow & & \downarrow \psi_{v,u} \\ u^*v_*(F_v) & \xrightarrow{u^*v_*(g_u)} & u^*v_*(G_v) \end{array} \qquad (2)$$

is commutative for all $u, v \in \mathfrak{U}$. The compositon is defined in an obvious way. We denote the category of $\mathfrak{U}$-comodules by $\mathfrak{U} - Comod$.

There is a natural functor $\mathcal{F}_X \longrightarrow \mathfrak{U} - Comod$ which assigns to any $M \in Ob\mathcal{F}_X$ the $\mathfrak{U}$-comodule $(M^\sim, \phi^M)$, where $M^\sim_u = u^*(M)$ and $\phi^M_{u,v} := v^*\eta_u(M) : v^*(M) \longrightarrow v^*u_*u^*(M)$.

1.0.8.2. Proposition. *Let $\mathfrak{U} = \{U \xrightarrow{u} X\}$ be a finite set of continuous morphisms of the base B.*

1) The following properties are equivalent:

(a) A pair of arrows $f, g : L \longrightarrow M$ of $\mathcal{F}_X$ has a kernel iff the pair $(u^*(f), u^*(g))$ has a kernel for all $u \in \mathfrak{U}$, and any functor u^*, $u \in \mathfrak{U}$, maps a kernel of (f, g) (if any) into a kernel of $(u^*(f), u^*(g))$.

(b) The functor $\mathcal{F}_X \longrightarrow \mathfrak{U} - Comod$ is an equivalence of categories.

2) If the equivalent conditions of 1) hold, then the cone of the functor morphisms

$$\begin{array}{ccc} Id_{\mathcal{F}_X} & \xrightarrow{\eta_u} & G_u \\ \eta_v\downarrow & & \downarrow \eta_v G_u \\ G_v & \xrightarrow{G_v\eta_u} & G_vG_u \end{array} \qquad u, v \in \mathfrak{U} \qquad (3)$$

is terminal. Here $G_u := u_*u^*$ and η_u is an adjunction morphism $Id_{\mathcal{F}_X} \longrightarrow G_u$.

Proof. 1) If $\mathfrak{U}$ consists of one morphism, the assertion 1) follows from the Barr-Beck theorem (cf. [ML], VI.7). To the family of functors $\{u^*|u \in \mathfrak{U}\}$, there corresponds a functor $\mathbf{u}^* : X \longrightarrow \prod_{(U \xrightarrow{u} X)\in\mathfrak{U}} \mathcal{F}_U$. If the category $\mathcal{F}_X$ has products, then the functor $\mathbf{u}^*$ has a right adjoint, and $\mathbf{u}^*$ satisfies the condition (a).

2) Suppose that $\mathfrak{U}$ consists of one morphism u. Then the square

$$\begin{array}{ccc} u^* & \xrightarrow{u^*\eta_u} & u^*G_u \\ u^*\eta_u\downarrow & & \downarrow u^*\eta_u G_u \\ u^*G_u & \xrightarrow{u^*G_u\eta_u} & u^*G_uG_u \end{array}$$

is cartesian. Since u^* reflects kernels of pairs of morphisms, it follows that the square

$$\begin{array}{ccc} Id_{\mathcal{F}_X} & \xrightarrow{\eta_u} & G_u \\ \eta_u \downarrow & & \downarrow \eta_u G_u \\ G_u & \xrightarrow{G_u \eta_u} & G_u G_u \end{array}$$

is cartesian. The general case can be reduced to the case of one morphism, as in 1) above. Details are left to the reader. ■

1.1. The category of 'spaces'. 'Spaces' are categories which are equivalent to small categories with respect to some universum, $\mathfrak{U}$. We define *a morphism* f from a 'space' $\mathcal{A}$ to a 'space' $\mathcal{B}$ as an isomorphy class of right exact functors from $\mathcal{B}$ to $\mathcal{A}$. Any functor $\mathcal{B} \longrightarrow \mathcal{A}$ from f will be called *an inverse image functor of* f. And once we made a choice of an inverse image functor, we shall denote it by f^*. The composition of morphisms is natural: $f \circ g = [g^* \circ f^*]$. Here $[u]$ means "all functors isomorphic to u". Thus defined *category of 'spaces'* shall be denoted by $\mathcal{R}Cat_{\mathfrak{U}}$ or simply by $\mathcal{R}Cat$.

The category $\mathcal{R}Cat$ is a base of a naturally defined fibered category which we denote by $\mathcal{R}Cat^{\sim}$ and will call the *fibered category of 'spaces'*.

1.1.1. Proposition. *(a) The category of 'spaces' $RCat$ has small colimits.*
(b) For any category C, the category $RCat/C$ has small colimits.

Proof. (a1) The coproduct of a family $\{A_i | i \in J\}$ is the product of categories $\prod_{i \in J} A_i$.

(a2) *The category $RCat$ has fibered coproducts, in particular it has cokernels of pairs of morphisms.*

In fact, let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be morphisms in $RCat$. Denote by W the category objects of which are triples (L, ϕ, M), where $L \in ObX$, $M \in ObY$, and ϕ an isomorphism $p^*(L) \longrightarrow q^*(M)$. Morphisms from (L, ϕ, M) to (L', ϕ', M') are pairs of morphisms $s : L \longrightarrow L'$, $t : M \longrightarrow M'$ such that the diagram

$$\begin{array}{ccc} p^*(L) & \xrightarrow{\phi} & q^*(M) \\ p^*(s) \downarrow & & \downarrow q^*(t) \\ p^*(L') & \xrightarrow{\phi'} & q^*(M') \end{array}$$

is commutative. The functors $W \longrightarrow X$, $(L, \phi, M) \longmapsto L$, and $W \longrightarrow Y$, $(L, \phi, M) \longmapsto M$, are right exact, hence can be regarded as inverse images of resp. morphisms $q_1 : X \longrightarrow W$ and $p_1 : Y \longrightarrow W$. It follows that $q_1 \circ p = p_1 \circ q$. Any pair of morphisms $X \xrightarrow{u} V \xleftarrow{v} Y$ such that $u \circ p = v \circ q$, defines uniquely up to isomorphism a functor $g^* : V \longrightarrow W$ which assigns to any $N \in ObV$ the triple $(u^*(N), \phi_{u,v}(N), v^*(N))$. Here $\phi_{u,v}$ is a functor isomorphism $p^*u^* \longrightarrow q^*v^*$, and $\phi_{u,v}(N)$ denotes its value on the object N. The functor g^* is an inverse image functor of a morphism $g : W \longrightarrow V$ uniquely defined by the equality $u \circ p = v \circ q$. Therefore W is a fibered coproduct of $X \xleftarrow{p} Z \xrightarrow{q} Y$.

It follows from (a1) and (a2) that $RCat$ has colimits of arbitrary small diagrams.

(b) Let $f : \coprod_{i\in J} X_i \longrightarrow C$ be a morphism with the inverse image functor $f^* := \prod_{i\in J} f_i^* : C \longrightarrow \prod_{i\in J} X_i$. For any object $(B, g : B \longrightarrow C)$ of the category $RCat/C$, the canonical map $RCat/C((\coprod_{i\in J} X_i, f), (B, g)) \longrightarrow \prod_{i\in J} RCat/C((X_i, f_i), (Y, g))$ is a bijection, hence $(\coprod_{i\in J} X_i, f)$ is a coproduct of the family $\mathcal{F}$ in the category $RCat/C$.

To finish the argument, it suffices to show that $RCat/C$ has fibered coproducts.

Let $(X, f) \overset{p}{\longleftarrow} (Z, h) \overset{q}{\longrightarrow} (Y, g)$ be morphisms in $RCat/C$. By definition, $f \circ p = h = g \circ q$, or, equivalently, $p^*f^* \simeq h^* \simeq q^*g^*$. In particular, we have a functorial isomorphism $\psi : p^*f^* \longrightarrow q^*g^*$, hence a functor $t^* : C \longrightarrow W = X \coprod_{p,q} Y,\ E \longmapsto (f^*(E), \psi(E), g^*(E))$. It follows that t^* is an inverse image functor of a morphism $t : W \longrightarrow C$ and the canonical morphisms $X \overset{q_1}{\longrightarrow} W \overset{p_1}{\longleftarrow} Y$ are morphisms over C, i.e. $f = t \circ q_1$ and $g = t \circ p_1$. This proves the assertion. ∎

A morphism $f : \mathcal{B} \longrightarrow \mathcal{A}$ such that f^* is a localization (i.e. a universal functor making invertible all arrows of $\Sigma_f := \{s \in Hom\mathcal{A} | f^*(s) \text{ is invertible}\}$) will be called by abuse of language a localization.

1.1.2. Proposition. *Let $A \overset{f}{\longrightarrow} C \overset{g}{\longleftarrow} B$ be morphisms of 'spaces' such that A and B are categories with finite colimits and f (i.e. f^*) is a localization. Then the fiber product $A \times_{(f,g)} B$ exists and the canonical projection $A \times_{(f,g)} B \longrightarrow B$ is a localization.*

Proof. Denote by Σ_{f^*} the class of morphisms of the category C the functor f^* sends to isomorphisms. Since f^* is right exact, Σ_f admits left fractions. Let $\Sigma_{f,g}$ be the minimal class of morphisms of the category B containing $g^*(\Sigma_f)$ and admitting left fractions. Let $q^* : B \longrightarrow B' := \Sigma_{f,g}^{-1}B$ be a localization at $\Sigma_{f,g}$. Since q^* is right exact, it can be regarded as an inverse image functor of a morphism, q. The composition $q^* \circ g^*$ maps Σ_f to isomorphisms, and f^* is a localization at Σ_f. Hence there exists a unique functor $p^* : A \longrightarrow B'$ such that $p^*f^* = q^*g^*$. Since the functor q^*g^* is right exact, p^* is right exact too. Thus we have a commutative square

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & C \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle g} \\ B' & \overset{q}{\longrightarrow} & B \end{array} \qquad (1)$$

The claim is that the square (1) is cartesian. In fact, let $A \overset{u}{\longleftarrow} B'' \overset{v}{\longrightarrow} B$ be morphisms such that $fu = gv$. Since v^* is a right exact functor, the class Σ_v of morphisms v^* maps to isomorphisms admits left fractions. Since v^*g^* being isomorphic to u^*f^* maps Σ_f to isomorphisms, the class Σ_v contains $\Sigma_{f,g}$. Therefore, $v^* = h^*q^*$ for a uniquely defined functor $h^* : B' \longrightarrow B''$. Since the functor v^* and the localization q^* are right exact, h^* is right exact too, hence it might be regarded as an inverse image functor of a morphism $h : B'' \longrightarrow B'$. Thus $h^*p^*f^* = h^*q^*g^* \simeq u^*f^*$. Since f^* is a localization, it follows from $h^*p^*f^* \simeq u^*f^*$ that $h^*p^* \simeq u^*$, i.e. $u = ph$. ∎

1.2. Monads and continuous morphisms. Fix a category C. Denote by $RCat_C$ the full subcategory of the category $RCat/C$ objects of which are continuous morphisms $f : X \longrightarrow C$. To any continuous morphism $f : X \longrightarrow C$, there corresponds a monad

$\mathbb{G}_f = (G_f, \mu_f)$ uniquely defined by the choice of the inverse and direct image functors, f^* and f_*, and an adjunction arrow $\epsilon_f : f^*f_* \longrightarrow Id_X$. Namely $G_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^* : G_f^2 \longrightarrow G_f$. The adjunction arrow $\eta_f : Id_C \longrightarrow f_*f^* = G_f$ is the unit of $\mathbb{G}_f$. One can check that to different choices of (f^*, f_*, ϵ_f) there correspond isomorphic monads.

1.2.1. Proposition. *For any category C with cokernels of reflexive pairs of morphisms, the canonical morphism of fibered categories $\mathfrak{Mod}C \longrightarrow RCat_2^{\sim C}$ is an equivalence.*

Proof. The fact follows from 1.0.7.2. ■

1.3. Coproducts in $RCat_C$.

1.3.1. Proposition. *Let $\mathcal{F} = \{f_i : X_i \longrightarrow C | i \in J\}$ be a family of morphisms. Suppose that C has products of sets of $|J|$ objects. If all morphisms f_i are continuous, then their coproduct in the category $RCat/C$ is continuous, hence it is a coproduct in the category $RCat_C$.*

Proof. (a) For any $(B, g : B \longrightarrow C)$, the canonical map

$$RCat/C((\coprod_{i\in J} X_i, f), (B, g)) \longrightarrow \prod_{i\in J} RCat/C((X_i, f_i), (Y, g))$$

is a bijection.

(b) Let f_{i*} be a direct image functor of the morphism f_i. Then $M = (M_i | i \in J) \longmapsto \prod_{i\in J} f_{i*}(M_i)$ is a direct image functor of f. ■

1.3.2. The monad associated with a coproduct. Let $\{f_i : X_i \longrightarrow C | i \in J\}$ be a family of continuous morphisms. Let f be their coproduct. Assume that the condition of 1.3.1 holds, hence f is continuous. The monad (f_*f^*, μ_f) corresponding to f is isomorphic to the product $\prod_{i\in J} \mathbb{G}_{f_i}$ of monads $\mathbb{G}_{f_i} = (f_{i*}f_i^*, \mu_{f_i})$ corresponding to morphisms f_i.

1.4. Glueing two 'spaces'.

1.4.1. Proposition. *Let $X \overset{p}{\longleftarrow} Z \overset{q}{\longrightarrow} Y$ be morphisms of 'spaces' such that q (i.e. q^*) is a localization (resp. a continuous localization). Then the canonical morphism $q_1 : X \longrightarrow X \coprod_{(p,q)} Y$ is a localization (resp. a continuous localization).*

Proof. (a) Denote by $\Sigma_{q_1^*}$ the class of all morphisms s of the category $X \coprod_{(p,q)} Y$ such that $q_1^*(s)$ is invertible. Note that $\Sigma_{q_1^*}$ consists of all morphisms $(s, t) : (L, \phi, M) \longrightarrow (L', \phi', M')$ such that s is an isomorphism, hence $t \in \Sigma_{q^*}$. Since q^* is a localization, for any $L \in ObX$, there exists an $M \in ObY$ such that $p^*(L) \simeq q^*(M)$. This defines a functor $X \longrightarrow \Sigma_{q_1^*}^{-1}(X \coprod_{(p,q)} Y)$ quasi-inverse to the canonical functor $\Sigma_{q_1^*}^{-1}(X \coprod_{(p,q)} Y) \longrightarrow X$, $(L, \phi, M) \longmapsto L$.

(b) Suppose the morphism q is continuous, i.e. there exists a direct image functor q_*. Fix adjunction morphisms $\eta_q : Id_Y \longrightarrow q_*q^*$ and $\epsilon_q : q^*q_* \longrightarrow Id_Z$. Since q^* is a localization, the morphism ϵ_q is an isomorphism. In particular, we have a functor $q_{1*} : X \longrightarrow X \coprod_{(p,q)} Y$, $L \longmapsto (L, \epsilon_q^{-1}(p^*(L)), q_*p^*(L))$. This functor is a right adjoint to the functor $q_1^* : (L, \phi, M) \longmapsto L$. The adjunction morphism $\epsilon_{q_1} : q_1^* \circ q_{1*} \longrightarrow Id_X$

is identical; the adjunction morphism $\eta_{q_1} : Id_{X\coprod_{(p,q)} Y} \longrightarrow q_{1*}q_1^*$ assigns to any object (L,ϕ,M) of the category $X\coprod_{(p,q)} Y$ the morphism

$$(id_L, q_*(\phi^{-1})\circ \eta_q(M)) : (L,\phi,M) \longrightarrow q_{1*}q_1^*(L) := (L, \epsilon_q^{-1}(p^*(L)), q_*p^*(L))$$

hence the assertion. ■

1.4.2. Proposition. *Let* $X \overset{p}{\longleftarrow} Z \overset{q}{\longrightarrow} Y$ *be morphisms of 'spaces' such that* q *and* p *(i.e.* q^* *and* p^**) are localizations. Then the square*

$$\begin{array}{ccc} Z & \overset{q}{\longrightarrow} & Y \\ p \downarrow & & \downarrow p_1 \\ X & \overset{q_1}{\longrightarrow} & X\coprod_{(p,q)} Y \end{array}$$

is cartesian.

Proof. Let $X \overset{u}{\longleftarrow} W \overset{v}{\longrightarrow} Y$ be morphisms such that $q_1\circ u = p_1\circ v$. In other words, there exists an isomorphism $\phi : u^*\circ q_1^* \overset{\sim}{\longrightarrow} v^*\circ p_1^*$. Let $s : M \longrightarrow M'$ be any morphism of Σ_{q^*}. Since p is a localization, there exists $L\in ObX$ and an isomorphism $\phi : p^*(L) \longrightarrow q^*(M)$. Consider the morphism $s' = (id_L, s) : (L,\phi,M) \longrightarrow (L, q^*(s)\circ\phi, M')$ of the category $X\coprod_{(p,q)} Y$. By definition of q_1^* and p_1^* (cf. the argument of 1.1.1), $q_1^*(s') = id_X$ and $p_1^*(s') = s$, hence $u^*\circ q_1^*(s') = id_W$ and $v^*\circ p_1^*(s') = v^*(s)$. Since $u^*\circ q_1^* \simeq v^*\circ p_1^*$, it follows that $v^*(s) = id_W$. Therefore the functor v^* maps Σ_q into invertible morphisms. Since q^* is a localization, there exists a unique functor $w^* : Y \longrightarrow W$ such that $v^* = w^*\circ q^*$. Since the functors q^* and v^* are right exact, the functor w^* is right exact, hence can be regarded as an inverse image functor of a morphism $w : Z \longrightarrow W$. Thus, the morphism v is uniquely represented as a composition $w\circ q$. Similarly, the morphism $u = u'\circ p$ for a uniquely defined morphism $u' : Z \longrightarrow W$, hence the assertion. ■

1.4.3. Almost affine localizations. We call a continuous morphism $f : X \longrightarrow Y$ *almost affine*, if its direct image functor is faithful and exact.

Note that f_* is exact iff it is right exact. The composition of almost affine morphisms is almost affine.

1.4.3.1. Proposition. *Let* $X \overset{p}{\longleftarrow} Z \overset{q}{\longrightarrow} Y$ *be morphisms of 'spaces' such that* q *(i.e.* q^**) is a localization. If* q *is almost affine, then the canonical morphism* $q_1 : X \longrightarrow X\coprod_{(p,q)} Y$ *is an almost affine localization.*

Proof. By the argument of 1.3.4.1(b), q_{1*} maps any object L of the category X into $(L, \epsilon_q^{-1}(p^*(L)), q_*p^*(L))$. By 1.3.4.1, q_{1*} is a right adjoint to a localization, hence (fully) faithful. By assumption, the functor q_* is right exact, hence q_*p^* is right exact which implies that q_{1*} is right exact. ■

1.5. The category of cospaces. We denote by $LCat$, or by $LCat_{\mathfrak{U}}$, the category objects of which are categories equivalent to small categories with respect to some universum, $\mathfrak{U}$. A *morphism* f from $\mathcal{A}$ to $\mathcal{B}$ as an isomorphy class of left exact functors from $\mathcal{B}$ to

$\mathcal{A}$. Any functor $\mathcal{B} \longrightarrow \mathcal{A}$ from f will be called *a direct image functor of* f. Once we made a choice of a direct image functor, we denote it by f_*. The composition of morphisms is defined by $f \circ g = [f_* \circ g_*]$. Here $[u]$ means 'all functors isomorphic to u'.

The category $LCat$ is a base of a naturally defined cofibered category which we denote by $LCat^{\sim}$ and will call the *cofibered category of 'cospaces'*.

We denote by $\mathcal{R}_c Cat$ the subcategory of the category $RCat$ formed by continuous morphisms and by $\mathcal{R}_c Cat^{\sim}$ the corresponding bifibered category. We have natural embedding $R_c Cat \longrightarrow RCat$ and the faithful functor $R_c Cat \longrightarrow LCat$ which maps any morphism to $f : X \longrightarrow Y$ to the isomorphy class of $[f_*] : X \longrightarrow Y$ of its direct image functor.

The functor $Cat \longrightarrow Cat$ which assigns to any small subcategory its dual and to any functor the corresponding dual functor induces an equivalence between the categories $RCat^{op}$ and $LCat$. Thus the dual versions of assertions about $RCat$ hold in $LCat$. For instance, $LCat$ is a category with limits, in particular with fiber products.

The existence of a limit or a colimit of a diagram $\mathfrak{D} : D \longrightarrow R_c Cat$ is usually a ruther subtle problem. The following trivial observation may be regarded as a starting point. If $X \longrightarrow \mathfrak{D}$ is a cone such that the image of this cone in $LCat$ is an initial cone, then the cone itself is initial, i.e. X is a limit of $\mathfrak{D}$ in $R_c Cat$.

1.6. Closed 'subspaces' and cosubspaces. For a morphism $f : X \longrightarrow Y$ of a category A, denote by Λ_f the class of all pairs of morphisms $u_1, u_2 : X \rightrightarrows V$ equalizing f. A morphism $f : Y \longrightarrow X$ is called a *strict monomorphism* if any morphism $g : Z \longrightarrow X$ such that $\Lambda_f \subseteq \Lambda_g$ has a unique decomposition $g = f \circ g'$. It follows that any strict monomorphism is a monomorphism. Suppose a morphism $f : Y \longrightarrow X$ is such that there exists a fibered coproduct $X \coprod_Y X$. Then f is a strict monomorphism iff it is a kernel of the coprojections $Y \rightrightarrows Y \coprod_X Y$. We denote the class of strict monomorphisms of the category C by $\mathfrak{M}_s(C)$, or by $\mathfrak{M}_s$. The class $\mathfrak{E}_s = \mathfrak{E}_s(C)$ of strict epimorphisms is defined dually. We say that an object Y is a *strict subquotient* of an object X if there exists a diagram $X \stackrel{m}{\longleftarrow} K \stackrel{e}{\longrightarrow} Y$, where m is a strict monomorphism and e is a strict epimorphism.

We call a full subcategory B of a category A *topologizing* if B contains all strict subquotients of any of its objects and limits and colimits of all finite diagrams $D \longrightarrow B$ taken in A (if any).

We call a full topologizing subcategory B of a category A a *closed 'subspace' of* A (resp. a *closed cosubspace of* A), if the inclusion functor $J : B \longrightarrow A$ has a left (resp. right) adjoint. Thus if B is a closed 'subspace' of A, the inclusion functor $J : B \longrightarrow A$ can be regarded as a direct image functor of an embedding $B \longrightarrow A$. Closed cosubspaces of A are exactly closed 'subspaces' of A^{op}.

We call a full subcategory B of A a *bisubspace of* A if it is both a closed 'subspace' and a closed cosubspace. Sometimes bisubspaces will be called *Zariski closed 'subspaces'*.

1.6.1. Example: closed 'subspaces' of the category of modules. Let A be the category $R-mod$ of left R-modules over a ring R. Then bisubspaces of A are in one-to-one correspondence with two-sided ideals of the ring R: to any two-sided ideal α of R, there corresponds the bisubspace $R/\alpha - mod$ – the full subcategory of $R - mod$ generated by all R-modules annihilated by the ideal α (see [R], III.6.4.1).

1.6.2. Example: closed cosubspaces of the category of modules. Fix a left ideal m of a ring R. Let $[R/m]$ denote the minimal cosubspace of the category $R - mod$ of left R-modules containing the module R/m. This cosubspace admits the following description.

For two left ideals, m and n, we write $m \leq n$ if there exists a finite subset x of elements of R such that the left ideal $(m : x) := \{r \in R| \ rx \subseteq n\}$ is contained in n. The cosubspace $[R/m]$ is a full subcategory of $R - mod$ objects of which are all R-modules M such that for any element z of M, $m \leq Ann(z)$. Here $Ann(z)$ denotes the annihilator of z.

Note that if the ideal m is two-sided, then $m \leq n$ iff $m \subseteq n$. In particular, if m is a two-sided ideal, the category $[R/m]$ coincides with the bisubspace $R/m - mod$ of Example 1.6.1.

2. Affine schemes.

2.1. Continuous and $\mathfrak{S}$-continuous functors. Fix a class of small categories $\mathfrak{S}$. For any two categories A and B (objects of Cat), denote by $Fun_{\mathfrak{S}}(A, B)$ the full subcategory of the category $Fun(A, B)$ of functors from A to B objects of which are all functors preserving colimits of functors from $\mathfrak{S}$.

2.1.1. $\mathfrak{S}_2$-continuous functors. A special case of this construction is the category $Fun_2(A, B)$ objects of which are functors preserving cokernels of reflexive pairs of arrows. The corresponding class consists of one category, $\mathfrak{S}_2$, with two objects, a, b, and six arrows: id_a, id_b, $e_i : a \longrightarrow b$, $h : b \longrightarrow a$, and $f_i : a \longrightarrow b$, $i = 1, 2$, such that $e_1^2 = e_2 \circ e_1 = e_1$, $e_2^2 = e_1 \circ e_2 = e_2$, $h \circ f_i = e_i$, $i = 1, 2$.

2.1.2. Right exact functors. Another special case is the category $Fun_{fin}(A, B)$ objects of which are right exact functors (i.e. functors which preserve colimits of all finite diagrams).

2.1.3. ω-continuous functors. The category $Fun_\omega(A, B)$ objects of which are functors preserving colimits of countable diagrams will also play a particular role in this work beginning from 2.6.

2.1.4. The categories $Cat_{\mathfrak{S}}$. For any three categories, A, B, C, the composition map

$$Fun_{\mathfrak{S}}(A, B) \times Fun_{\mathfrak{S}}(B, C) \longrightarrow Fun(A, C), \quad (F, G) \longmapsto G \circ F,$$

takes values in $Fun_{\mathfrak{S}}(A, C)$. This defines a subcategory $Cat_{\mathfrak{S}}$ of the category Cat. In particular, we have the subcategories Cat_2, Cat_{fin}, and Cat_ω.

2.1.5. Continuous functors. We call a functor $F : A \longrightarrow B$ *continuous* if it has a right adjoint. The full subcategory of $Fun(A, B)$ objects of which are continuous functors will be denoted by $Fun_c(A, B)$. Clearly the composition of continuous functors is a continuous functor. Hence we have a subcategory Cat_c of the category Cat.

2.2. Proposition. *Let $f : X \longrightarrow Y$ be a continuous morphism. Suppose the category X has cokernels of reflexive pairs of arrows and a direct image functor f_* of f preserves cokernels of reflexive pairs and reflects isomorphisms. Then*

(a) The functor f_* preserves colimits of functors from $\mathfrak{S}$ iff the functor $f_* f^*$ has this property.

(b) The functor f_* has a right adjoint iff f_*f^* has a right adjoint.

Proof. (a) If f_* preserves colimits of functors from $\mathfrak{S}$, then f_*f^* enjoys this property (without any additional conditions), since f^* preserves all colimits.

Conversely, suppose f_*f^* preserves colimits of functors from $\mathfrak{S}$. Then the forgetful functor $\mathbb{G}_f - mod \longrightarrow Y$, where $\mathbb{G}_f$ is the monad (f_*f^*, μ_f), has the same property. By the Barr-Beck theorem, the canonical functor $K_f : X \longrightarrow \mathbb{G}_f - mod,\ M \longmapsto (f_*(M), \mu_f(M))$ is an equivalence, in particular it preserves all colimits. The composition of the functor K_f with the forgetful functor coincides with f_*, hence the assertion.

(b) If f_* is a right adjoint, $f^!$, then $f_*f^!$ is a right adjoint to the functor f_*f^*.

Suppose $\mathbb{F} = (F, \mu)$ is a monad on Y such that the functor F has a right adjoint, $F^!$. Then the forgetful functor $\mathbb{F} - mod \longrightarrow Y$ has a right adjoint ([R3], 4.4.1). Since f_* is the composition of the forgetful functor $\mathbb{G}_f - mod \longrightarrow Y$ and the equivalence $K_f : X \longrightarrow \mathbb{G}_f - mod,\ M \longmapsto (f_*(M), \mu_f(M))$, it has a right adjoint too. ■

2.3. Relative affine and $\mathfrak{S}$-affine schemes. For a class of small categories $\mathfrak{S}$, a morphism $f : X \longrightarrow Y$ is called $\mathfrak{S}$-*affine* if the object (X, f) of the category $RCat/Y$ is isomorphic to the object $((F, \mu) - mod_Y, f_Y)$ for some monad (F, μ) on Y such that F is a $\mathfrak{S}$-continuous functor. Here f_Y is the canonical morphism $(F, \mu) - mod_Y \longrightarrow Y$ with a direct image functor equal to the forgetful functor.

The object (X, f), where $f : X \longrightarrow Y$ is a $\mathfrak{S}$-affine morphism, will be called an $\mathfrak{S}$-*affine Y-scheme*. We denote by $Aff_{\mathfrak{S}}/Y$ the full subcategory of the category $RCat/Y$ objects of which are $\mathfrak{S}$-affine Y-schemes and by $Aff_{\mathfrak{S}}^{\sim}/Y$ the corresponding fibered category.

2.3.1. Almost affine and ω-affine relative schemes. We single out two important special cases of $\mathfrak{S}$-affine relative schemes:

A morphism $f : X \longrightarrow Y$ is called *almost affine* (resp. ω-*affine*), if (X, f) is isomorphic to $((F, \mu) - mod_Y, f_Y)$ for some monad (F, μ) such that F is a right exact functor (resp. F preserves colimits of countable diagrams). The object (X, f) of $RCat/Y$ will be called resp. *almost affine*, or ω-*affine* Y-scheme. Thus we have the category Aff_{fin}/Y of almost affine Y-schemes and the category Aff_{ω}/Y of ω-affine Y-schemes and the corresponding fibered categories, resp. $Aff_{fin}^{\sim}/Y$ and $Aff_{\omega}^{\sim}/Y$.

2.3.2. Affine relative schemes. A morphism $f : X \longrightarrow Y$ is called *affine*, if (X, f) is isomorphic to $((F, \mu) - mod_Y, f_Y)$ for some monad (F, μ) such that F is continuous, i.e. has a right adjoint. An object (X, f) of the category $RCat/Y$ will be called an *affine Y-scheme* if the morphism $f : X \longrightarrow Y$ is affine. The full subcategory of $RCat/Y$ objects of which are affine Y-schemes will be denoted by Aff/Y.

It follows from 2.1.4 and 2.1.5 that the composition of affine (resp. $\mathfrak{S}$-affine) morphisms is affine (resp. $\mathfrak{S}$-affine).

2.3.3. Proposition. *Let $(X, f) \in ObRCat/Y$. Suppose a category X has cokernels of reflexive pairs of morhisms. Then (X, f) is a $\mathfrak{S}$-affine (resp. almost affine, resp. affine) Y-scheme iff the morphism f is continuous and its direct image functor preserves cokernels of reflexive pairs of morphisms and colimits of functors from $\mathfrak{S}$ (resp. is exact, resp. has a right adjoint) and reflects isomorphisms.*

Proof. The assertion follows from 2.2 and the Barr-Beck theorem (cf. [ML], VI.7). ∎

2.4. Relative $\mathfrak{S}$-affine schemes and $\mathfrak{S}$-monads. Fix a class of small categories $\mathfrak{S}$. For a category Y, denote by $\mathfrak{Mon}_{\mathfrak{S}}Y$ the full subcategory of the category of $\mathfrak{Mon}Y$ of monads on Y objects of which are monads (F,μ) such that F preserves cokernels of reflexive pairs of arrows and colimits of functors from $\mathfrak{S}$. As particular cases, we have the subcategory $\mathfrak{Mon}_{fin}Y$ of *right exact monads on* Y objects of which are monads (F,μ) such that the functor F is right exact, and the subcategory $\mathfrak{Mon}_{\omega}Y$ of ω*-monads on* Y which are monads (F,μ) such that F preserves colimits of countable diagrams. Finally, we denote by $\mathfrak{Mon}_cY$ the full subcategory of $\mathfrak{Mon}Y$ object of which are *continuous monads on* Y, that is monads (F,μ) with F having a right adjoint.

Assume that Y has cokernels of reflexive pairs of arrows. Then, by 1.0.6.1, we have natural functors

$$\mathbf{Spec}_{\mathfrak{S}}/Y : (\mathfrak{Mon}_{\mathfrak{S}}Y)^{op} \longrightarrow Aff_{\mathfrak{S}}/Y. \tag{1}$$

$$\mathbf{Spec}_c/Y : (\mathfrak{Mon}_cY)^{op} \longrightarrow Aff/Y. \tag{2}$$

which are compositions of natural pseudo-functors respectively

$$\mathbf{Spec}_{\mathfrak{S}}^{\sim}/Y : (\mathfrak{Mon}_{\mathfrak{S}}Y)^{op} \longrightarrow Aff_{\mathfrak{S}}^{\sim}/Y. \tag{1'}$$

$$\mathbf{Spec}_c^{\sim}/Y : (\mathfrak{Mon}_cY)^{op} \longrightarrow Aff^{\sim}/Y. \tag{2'}$$

with the corresponding projections.

We single out two special cases of (1):

$$\mathbf{Spec}_{fin}/Y : (\mathfrak{Mon}_{fin}Y)^{op} \longrightarrow Aff_{fin}/Y. \tag{3}$$

and

$$\mathbf{Spec}_{\omega}/Y : (\mathfrak{Mon}_{\omega}Y)^{op} \longrightarrow Aff_{\omega}/Y. \tag{4}$$

It follows from 1.0.7.1 that the functors $\mathbf{Spec}_{\mathfrak{S}}/Y$ and $\mathbf{Spec}_c/Y$ are full. They are not faithful in general.

2.5. Affine and almost affine k-schemes and k-'spaces'. If C is the category $k-mod$ of left modules over an associative ring k, we shall say *affine* (resp. *almost affine*, resp. $\mathfrak{S}$-*affine*) k-*schemes* instead of *affine* (resp. *almost affine*, resp. $\mathfrak{S}$-*affine*) C-*schemes.*

Fix a pseudo-functor

$$(RCat)^{op} \longrightarrow Cat, \quad ((X,f) \xrightarrow{h} (Y,g)) \longmapsto (Y \xrightarrow{h^*} X); \; c_{h,t} : t^*h^* \xrightarrow{\sim} (ht)^* \tag{1}$$

We associate with (1) the fibered category $(RCat/C)^{\sim} \xrightarrow{\pi} RCat/C$ (see 1.0) and a natural quasi-coherent presheaf (i.e. a section of the fibered category $(RCat/C)^{\sim} \longrightarrow RCat/C$ taking values in the category of quasi-coherent morphisms, cf. 1.0.2)

$$\mathcal{O} : RCat/C \longrightarrow (RCat/C)^{\sim}$$

which assigns to any object (X,f) of $RCat/C$ the object $(X,f^*(k))$ of the category $(RCat/C)^\sim$ and to any morphism $h:(X,f)\longrightarrow(Y,g)$ the morphism

$$(h, c_{g,h}(k) : h^*(g^*(k)) \longrightarrow f^*(k)).$$

Fix an object $(\mathcal{A},f)$ of $RCat/C$ such that $f:\mathcal{A}\longrightarrow C$ is a continuous morphism. Then it is defined uniquely up to isomorphism by the object $\mathcal{O}=f^*(k)$.

In fact, we have functorial isomorphisms $\mathcal{A}(f^*(k),X)\simeq\mathcal{C}(k,f_*(X))\simeq f_*(X)$ which shows that the direct image functor f_* of f is naturally isomorphic to the functor $X\longmapsto\mathcal{A}(f^*(k),X)$. Therefore the inverse image functor f^* (representing f) is defined uniqely up to isomorphism (being a left adjoint to the functor f_*) by the object $f^*(k)$. Note that since f^* preserves colimits, there exists a coproduct of any set of copies of $\mathcal{O}=f^*(k)$.

2.5.0. Note. Consider the category Cat_* objects of which are pairs $(\mathcal{A},\mathcal{O})$, where $\mathcal{A}$ is a category and $\mathcal{O}$ is an object of $\mathcal{A}$. Morphisms from $(\mathcal{A},\mathcal{O})$ to $(\mathcal{A}',\mathcal{O}')$ are pairs (f^*,ϕ), where f^* is a right exact functor from $\mathcal{A}$ to $\mathcal{A}'$ and ϕ is an isomorphism from $f^*(\mathcal{O}')$ to $\mathcal{O}$. Suppose that $(\mathcal{A},\mathcal{O})$ is an object of the category Cat_* such that the category $\mathcal{A}$ is abelian and there exists a coproduct of any set of copies of $\mathcal{O}$. Then the functor $M\longmapsto\mathcal{A}(\mathcal{O},M)$ from $\mathcal{A}$ to the category $K-mod$, where $K=\mathcal{A}(\mathcal{O},\mathcal{O})^o$, is a direct image of a continuous morphism from $\mathcal{A}$ to $K-mod$ ([BD], Proposition 6.6.23).

Now fix an additive category $\mathcal{A}$ and a continuous morphism $f:\mathcal{A}\longrightarrow C=k-mod$. And set $\mathcal{O}=f^*(k)$. The functor f_* is faithful iff $\mathcal{O}$ is a generator of the category $\mathcal{A}$. In this case, $\mathcal{A}$ has a structure of a k-linear category.

Since $f_*\simeq\mathcal{A}(\mathcal{O},-)$, the functor f_* is exact iff $\mathcal{O}$ is a projective object.

Thus f is almost affine iff $\mathcal{O}$ is a projective generator. Finally, f is affine iff $\mathcal{O}$ is a projective generator of finite type.

2.5.1. Proposition. *(a) For any continuous morphism $f:\mathcal{A}\longrightarrow C=k-mod$, there is a canonical functor morphism*

$$\psi_f : \mathcal{A}(\mathcal{O},\mathcal{O})^o\otimes_k \longrightarrow \mathcal{A}(\mathcal{O}, f^*-) \tag{1}$$

such that $\psi_f(V)$ is an isomorphism for any free k-module V of finite type.

(b) If f is almost affine, then $\psi_f(V)$ is an isomorphism for any finitely presented k-module V. *In particular, if k is left noetherian, then $\psi_f(V)$ is an isomorphism for any finitely generated k-module* V.

(c) The morphism ψ_f is an isomorphism if and only if f is affine.

(d) The morphism f is affine if and only if the functor

$$\mathcal{A}(\mathcal{O},-):\mathcal{A}\longrightarrow\mathcal{A}(\mathcal{O},\mathcal{O})^o-mod$$

is an equivalence of categories.

Proof. *(a)* For any additive functor $F:k-mod\longrightarrow k-mod$, the module $F(k)$ has a natural k-bimodule there is a canonical functor morphism $\psi_F:F(k)\otimes_k\longrightarrow F$ (see for instance, [Bass], Ch.I). Recall that, for any k-module V, the morphism $\psi_F(V)$ is the image of id_V with respect to the composition

$$\begin{array}{ccc} \mathrm{Hom}_k(V,V) & \longrightarrow & \mathrm{Hom}_k(V,\mathrm{Hom}_k(k,V)) \\ & & \downarrow \\ \mathrm{Hom}_k(F(k)\otimes_k V, F(V)) & \longleftarrow & \mathrm{Hom}_k(V,\mathrm{Hom}_k(F(k),F(V))) \end{array}$$

Since $\psi_F(k)$ is an isomorphism and the functor F is additive, $\psi_F(V)$ is an isomorphism for any free k-module V of finite rank.

(b) If the functor F is right exact, i.e. if it preserves cokernels, $\psi_F(V)$ is an isomorphism for any finitely presented object V, since finitely presented objects are exactly cokernels of morphisms between free objects of finite rank.

(c) The morphism ψ_F is an isomorphism iff the functor F preserves arbitrary colimits (or, equivalently, has a right adjoint).

The assertions *(a)–(c)* of the lemma are just specializations of these facts for the functor $f_* \circ f^* \simeq \mathcal{A}(\mathcal{O}, f^*-)$.

(d) If $f : \mathcal{A} \longrightarrow k - mod$ is almost affine, the canonical functor $\mathcal{A} \longrightarrow \mathbb{G}_f - mod$ is an equivalence of categories. The assertion *(c)* implies that if (and only if) f is affine, the monad $\mathbb{G}_f$ is naturally isomorphic to the monad $(\mathcal{A}(\mathcal{O},\mathcal{O})\otimes_k, m)$, where m is induced by the multiplication in $\mathcal{A}(\mathcal{O},\mathcal{O})^o$. The category $(\mathcal{A}(\mathcal{O},\mathcal{O})\otimes_k, m) - mod$ is isomorphic to the category of left modules over $\mathcal{A}(\mathcal{O},\mathcal{O})^o$. ■

2.5.2. Remark. The analysis above shows that when the ring k is commutative, affine schemes over $C = k - mod$ are affine schemes in the sense of M. Artin and J.J. Zhang [AZ]. ■

2.6. Product of ω-affine relative schemes.

2.6.1. The free product of continuous morphisms. Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be continuous morphisms with direct image functors resp. f_* and g_*. Denote by $A \star_{f,g} B$ the category objects of which are triples (X, ϕ, Y), where $X \in ObA$, $Y \in ObB$, and ϕ is an isomorphism $f_*(X) \longrightarrow g_*(Y)$. Morphisms between triples are defined in an obvious way. There are canonical functors $p_* : A \star_{f,g} B \longrightarrow A$, $(X, \phi, Y) \longmapsto X$ and $q_* : A \star_{f,g} B \longrightarrow B$, $(X, \phi, Y) \longmapsto Y$. It follows that $f_*p_* \simeq g_*q_*$.

2.6.1.1. Remark. Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be continuous morphisms. In terms of 1.5, the category $A \star_{f,g} B$ is a fiber product of the image $A \stackrel{[f_*]}{\longrightarrow} C \stackrel{[g_*]}{\longleftarrow} B$ of the diagram $A \stackrel{f}{\longrightarrow} C \stackrel{g}{\longleftarrow} B$ in the category $LCat$. If the canonical functor π_* : $A \star_{f,g} B \longrightarrow C$, $(X, \phi, Y) \longmapsto f_*(X)$ has a left adjoint, that is π_* is a direct image functor of a continuous morphism $\pi : A \star_{f,g} B \longrightarrow C$, then this morphism is a product of $A \stackrel{f}{\longrightarrow} C$ and $B \stackrel{g}{\longrightarrow} C$, or, in other words, a fiber product of A and B over C (cf. the end of 1.5).

Note however, that in spite of suggestive notations, the functors p_* and q_* do not necessarily have left adjoints.

2.6.1.2. Lemma. *Let F, G be two monads in a category C. Set $A = F - mod$, $B = G - mod$, and let $f : F - mod \longrightarrow C$, $g : G - mod \longrightarrow C$ be the corresponding* *canonical morphisms. Then the category by $A\star_{f,g} B$ is equivalent to the category $\mathfrak{M}_{f,g}$ objects of which are triples (m_F, M, m_G), where $M \in ObC$ and $m_F : F(M) \longrightarrow M$ and $m_G : G(M) \longrightarrow M$ are structures of resp. an F-module and a G-module on M. Morphisms from (m_F, M, m_G) to (m'_F, M', m'_G) are morphisms $M \longrightarrow M'$ which are morphisms of F-modules and G-modules. In other words, $A\star_{f,g} B$ is equivalent to its full subcategory objects of which are triples (X, id, Y).*

Proof. The functor quasi-inverse to the canonical embedding $\mathfrak{M}_{f,g} \longrightarrow A\star_{f,g} B$ assigns to any object (X,ϕ,Y) of the category $A\star_{f,g} B$ the module $(m_X, f_*(X), \phi^{-1}\circ m_Y G\circ\phi)$. Here m_X is the F-module structure of $X = (m_X, f_*(X))$, and m_Y is the G-module structrure of $Y = (m_Y, g_*(Y))$. ■

2.6.2. The free product of monads.

2.6.2.1. Lemma. *Let (Φ,ϕ) be a monoidal functor from $D^\sim = (D,\otimes,...)$ to $End^\sim C = (EndC, \circ, Id_C)$. Suppose there exists a colimit, F_Φ, of the functor Φ and the functors $\Phi(X)$, $X \in ObD$, preserve colimits of functors from D (say continuous). Then F_Φ has a canonical structure of a monad, $m_\phi : F_\Phi^2 \longrightarrow F_\Phi$.*

If Φ takes values in the subcategory of continuous functors and C is a category with limits, then the monad (F_Φ, m_ϕ) (i.e. the functor F_Φ) is continuous.

Proof. For any two objects, X and Y of the category D, denote by $m_{X,Y}$ the composition of $\phi_{X,Y} : \Phi(X)\circ\Phi(Y) \longrightarrow \Phi(X\otimes Y)$ and the canonical morphism $\Phi(X\otimes Y) \longrightarrow F_\Phi$. If $\Phi(X)$ preserves colimits, the morphisms $\{m_{X,Y} | Y \in ObD\}$ define a morphism $m_X : \Phi(X)\circ F_\Phi \longrightarrow F_\Phi$. The family of morphisms $\{m_X | X \in ObD\}$ defines a morphism $m_\phi : F_\Phi^2 \longrightarrow F_\Phi$ which is a monad structure on F_Φ.

If each functor $\Phi(X)$ has a right adjoint $\Phi(X)^\wedge$, and there exists a limit of the family of functors $\{\Phi(X)^\wedge | X \in ObD\}$, then this limit is a right adjoint to the functor F_Φ. ■

2.6.2.2. A monoidal functor associated with two monads. Let F, G be two monads with the units resp. η_F and η_G and the multiplications μ_F and μ_G. Denote by $D_{F,G}$ the category objects of which are 'monomials' $F^{i_1}G^{j_1}...F^{i_k}G^{j_k}$ in G and F, $i_k, j_k \geq 0$. We set $F^0 = G^0 = \mathbf{1}$ and $\mathbf{1}\cdot\mathbf{1} = \mathbf{1}$. Morphisms of $D_{F,G}$ are identical morphisms and those obtained by composing morphisms of the form $X\eta_G Y$, $X\eta_F Y$, $X\mu_G Y$, $X\mu_F Y$. Here X, Y are monomials in F and G, and $\mu_F : F^2 \longrightarrow F$, $\eta_F : \mathbf{1} \longrightarrow F$, $\mu_G : G^2 \longrightarrow G$, $\eta_G : \mathbf{1} \longrightarrow G$, are morphisms such that $\mu_F \circ F\eta_F = id_F = \mu_F\circ\eta_F F$ and $\mu_G\circ G\eta_G = id_G = \mu_G\circ\eta_G G$.

We turn $D_{F,G}$ into a strict monoidal category with the unit object $\mathbf{1}$ ('strict' means that the associativity constraint and the left and right multiplications by the unit object are identical morphisms) by setting

$$(F^{i_1}G^{j_1}...F^{i_k}G^{j_k})\otimes(F^{i'_1}G^{j'_1}...F^{i'_k}G^{j'_k}) := F^{i_1}G^{j_1}...F^{i_k}G^{j_k}F^{i'_1}G^{j'_1}...F^{i'_k}G^{j'_k}$$

and defining $\otimes$ on the morphisms in an obvious way.

There is a natural strict monoidal functor $\Phi = (\Phi, id)$ from $D_{F,G}$ to the category $EndC^\sim = (EndC, \circ)$ assigning to any object of the category $D_{F,G}$ the corresponding functor $C \longrightarrow C$.

2.6.2.3. Proposition. *(a) Let C be a category with colimits of countable diagrams. And let (F,μ) and (G,ν) be monads in the category C such that the functors F and G*

preserve colimits of countable diagrams. Then there exists a monad, $F \star G$, associated with the monoidal functor $\Phi, \mu : D_{F,G} \longrightarrow End^{\sim}C$.

(b) The monad $F \star G$ preserves countable colimits, in particular it is right exact.

(c) If the category C has countable limits and the monads F and G are continuous, then the monad $F \star G$ is continuous.

Proof. (a) If the category C has countable (co)limits, then the category $EndC$ has countable (co)limits. Therefore the assertion follows from 2.6.2.1.

(b) If the monads F and G preserve countable colimits, then all compositions of F and G have this property, hence $F \star G$, being a colimit of such compositions preserves countable colimits.

(c) The assertion follows from (the argument of) 2.6.2.1. ■

The monad $F \star G$ of Proposition 2.6.2.3 will be called the *free product*, or the *star-product* of the monads F and G.

2.6.2.4. Proposition. *Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be ω-affine schemes over C and $\mathbb{G}_f$, $\mathbb{G}_g$ associated monads. Suppose C is a category with countable colimits. Then $\mathbb{G}_f \star \mathbb{G}_g$ exists and is an ω-continuous monad.*

The canonical functor $A \star_{f,g} B \longrightarrow C$ is a direct image of an ω-affine morphism isomorphic to $\mathbb{G}_f \star \mathbb{G}_g - mod \longrightarrow C$. The ω-affine scheme $A \star_{f,g} B \longrightarrow C$ is a product of ω-affine schemes (A, f) and (B, g) in the category R_cCat/C of bispaces over C (cf. 1.5).

Proof. By Proposition 2.6.2.3, the monad $\mathbb{G}_f \star \mathbb{G}_g(M)$ is continuous, hence the morphism $\mathbb{G}_f \star \mathbb{G}_g - mod \longrightarrow C$ (and isomorphic to it $A \star_{f,g} B \longrightarrow C$) is affine. The assertion follows now from 2.6.2.4 and 2.2.5 (see the argument of 2.5.2.4). ■

2.6.2.4.1. Corollary. *Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be affine schemes over C and $\mathbb{G}_f$, $\mathbb{G}_g$ associated monads. Suppose C is a category with countable colimits and limits. Then $\mathbb{G}_f \star \mathbb{G}_g$ exists and is a continuous monad.*

The canonical functor $A \star_{f,g} B \longrightarrow C$ is a direct image of an affine morphism isomorphic to $\mathbb{G}_f \star \mathbb{G}_g - mod \longrightarrow C$. The affine scheme $A \star_{f,g} B \longrightarrow C$ is a product of affine schemes (A, f) and (B, g) in the category R_cCat/C of bispaces over C.

2.6.3. The free product with a localization.

2.6.3.1. Proposition. *Suppose C is a category with countable colimits. Let $A \xrightarrow{f} C$ be an ω-affine morphism, and let $B \xrightarrow{g} C$ be an ω-affine localization. Then the canonical morphism $A \star_{f,g} B \xrightarrow{p} A$ is an ω-affine localization.*

Proof. Let $\mathbb{G} = (G, \mu_G)$ be a monad in C. Suppose the morphism $g : \mathbb{G} - mod \longrightarrow C$ (i.e. its inverse image functor g^*) is a localization. This is equivalent to the following condition: the direct image functor $g_* : (M, m) \longmapsto M$ is fully faithful.

The category $\mathbb{G} \star \mathbb{F}$-modules is equivalent to the category $A \star B$ of triples (ν, M, m), where (M, m) is a $\mathbb{G}$-module and (M, ν) is a $\mathbb{F}$-module. The direct image functor p'_* of the canonical morphism $p' : \mathbb{G} \star \mathbb{F} - mod \longrightarrow \mathbb{F} - mod$ assigns to any triple (ν, M, m) the $\mathbb{F}$-module (M, ν). Clearly the functor p'_* is faithful. Since the functor $g_* : (M, m) \longmapsto M$ is full, the functor p'_* is full. The assertion follows now from 2.6.2.4. ■

2.6.3.2. Corollary. *Suppose C is a category with countable colimits and limits. Let $f : A \longrightarrow C$ be an affine morphism, and let $g : B \longrightarrow C$ be an affine localization. Then the canonical morphism $p : A \star_{f,g} B \longrightarrow A$ is an affine localization.*

2.6.3.3. Example. Suppose C is the category $R - mod$ of left modules over an associative ring R. Let $f : X \longrightarrow C$ and $g : Y \longrightarrow C$ be affine morphisms. This means that $X \simeq A - mod$, $Y \simeq B - mod$, and there are ring morphisms $R \longrightarrow A$ and $R \longrightarrow B$ corresponding to the morphisms resp. f and g. The morphism g is an affine localization iff the following conditions hold:

(a) B is flat as a right R-module, i.e. the functor $B \otimes_R$ is left exact;

(b) the canonical morphism $B \otimes_R B \longrightarrow B$ induced by the multiplication on B is an isomorphism.

It follows from 2.6.3.2 that $(A \star_R B)$ is a flat right A-module and the morphism $(A \star_R B) \otimes_A (A \star_R B) \longrightarrow A \star_R B$ induced by the multiplication is an isomorphism.

2.6.4. Faithfully flat morphisms. A morphism $\phi : (X, f) \longrightarrow (S, h)$ in $RCat_C$ is called flat (resp. faithful, resp. faithfully flat) if a corresponding inverse image functor is exact (resp. faithful, resp. faithful and exact).

2.6.4.1. Lemma. *Suppose C is a category with countable colimits.*

(a) Let $(X, f) \stackrel{\phi}{\longrightarrow} (S, h) \stackrel{\psi}{\longleftarrow} (Y, g)$ be morphisms in Aff/S such that ϕ is faithful. Then the projection $\pi : (X, f) \times_{(\phi,\psi)} (Y, g) \longrightarrow (Y, g)$ is faithful.

(b) If both morphisms ϕ and ψ are flat, then the projection π is flat.

Proof. (i) Suppose first that $(S, h) = (C, id)$. Then $(X, f) \times_{(\phi,\psi)} (Y, g) \simeq \mathbb{G}_f \star \mathbb{G}_g$ and the π is the natural morphism $\mathbb{G}_f \star \mathbb{G}_g - mod \longrightarrow \mathbb{G}_g - mod$ having $\mathbb{G}_f \star \mathbb{G}_g \otimes_{\mathbb{G}_g}$ as an inverse image functor.

(a) Since the functor f_* is faithful, the functor f^* is faithful iff $G_f := f_* f^*$ is faithful. It follows from 2.6.2.3 that the composition of this functor with the forgetful functor $p_* : \mathbb{G}_f \star \mathbb{G}_g - mod \longrightarrow C$ maps a $\mathbb{G}_g$-module (M, m) into $G_f(M) \oplus G'(M)$ for a certain functor G'. Since the functor G_f is faithful, the functor $G_f \oplus G'$ is faithful.

(b) If both functors g^* and f^* are exact, then by 2.6.2.3 the functor G_p is exact.

(ii) Replacing the category C by the category $S \simeq \mathbb{G}_h - mod$ and $\mathcal{A}$ by the corresponding category of $\mathbb{G}_h$-bimodules, we reduce the assertion to the case $(S, h) = (C, id)$. Details are left to the reader. ∎

3. Affine schemes in a monoidal category. Quasi-coherent modules.

Affine schemes in a monoidal category $\mathcal{A}^\sim$ are defined as the category opposite to the category of algebras (in the terminology of [ML] *monoids*) in $\mathcal{A}^\sim$. A relation with 'spaces over S' appeares when we have an action of the monoidal category $\mathcal{A}^\sim$ on the category S. Taking as $\mathcal{A}^\sim$ the monoidal subcategory $End^\sim_{\mathfrak{S}} S$ of endofunctors preserving colimits of the type $\mathfrak{S}$, or the subcategory $End^\sim_{c} S$ of continuous endofunctors (cf. 2.3), we recover back $\mathfrak{S}$-affine and affine schemes over S.

3.0. Preliminaries: algebras and (bi)modules in monoidal categories. Fix a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, 1, a, l, r)$. Here $\mathcal{A}$ is a category, $\otimes$ is a functor from $\mathcal{A} \times \mathcal{A}$ to

$\mathcal{A}$, a is a functor isomorphism $\otimes\circ(Id_{\mathcal{A}}\times\otimes) \longrightarrow \otimes\circ(\otimes\times Id_{\mathcal{A}})$ (*associativity constraint*), and $l : Id_{\mathcal{A}} \longrightarrow 1\otimes Id_{\mathcal{A}}$, $\quad r : Id_{\mathcal{A}} \longrightarrow Id_{\mathcal{A}}\otimes 1$ are functor isomorphisms compatible with the associativity constraint a. An *algebra* (or *monoid*) in $\mathcal{A}^\sim$ is a pair (R, μ) where $R \in Ob\mathcal{A}$ and μ is a morphism $R\otimes R \longrightarrow R$ such that $\mu\circ(\mu\otimes id_R)\circ a_{R,R,R} = \mu\circ(id_R\otimes\mu)$. The *unit* of an algebra (R, μ) is a morphism $\eta : 1 \longrightarrow R$ such that $\mu\circ\eta\otimes id_R\circ l_R = id_R = \mu\circ id_R\otimes\eta\circ r_R$. The unit (if it exists) is unique. We assume that all algebras considered here are unital. Algebras in $\mathcal{A}^\sim$ form a category which we denote by $Alg\mathcal{A}^\sim$.

A left module over an algebra (R, μ) is a pair (M, m), where $M \in Ob\mathcal{A}$, m is a morphism $R\otimes M \longrightarrow M$ such that $m\circ id_R\otimes m = m\circ\mu\otimes id_M\circ a_{R,R,M}$ and $m\circ\eta\otimes id_M = id_M$. Left modules over $R^\sim = (R, \mu)$ form a category $R^\sim - mod$.

The category $mod - R^\sim$ of right $R^\sim$-modules is defined in an obvious way. Note that right modules are just left modules in the opposite monoidal category. A triple (m, M, m'), where (m, M) and (M, m') are resp. left and right $R^\sim$-modules is called an $R^\sim$-*bimodule* if $m \circ id_R \otimes m' = m' \circ m \otimes id_R \circ a_{R,M,R}$.

Suppose that for any $X \in Ob\mathcal{A}$, the functor $X \longmapsto X \otimes -$ preserves cokernels of reflexive pairs of arrows. Then there is a well defined functor $\otimes_R : mod{-}R^\sim\times R^\sim{-}mod \longrightarrow \mathcal{A}$ which assigns to any pair of resp. right and left $R^\sim$-modules $(M, m), (\nu, N)$ the cokernel of the pair of morphisms $id_M\otimes\nu, m\otimes\nu\circ a_{M,R,N} : M\otimes(R\otimes N) \longrightarrow M\otimes N$. The functor $\otimes_R$ induces a structure of a monoidal category on the category $R^\sim - bimod$ of $R^\sim$-bimodules.

Let β be a symmetry of the monoidal category $\mathcal{A}^\sim$. An algebra $R^\sim = (R, \mu)$ is called β-*commutative* (or *commutative* if β is fixed) if $\mu \circ \beta_{R,R} = \mu$. The full subcategory of $Alg\mathcal{A}^\sim$ formed by β-commutative algebras will be denoted by $Alg_\beta\mathcal{A}^\sim$.

For any β-commutative algebra $R^\sim$, the map $(m, M) \longmapsto (m, M, m \circ \beta_{M,R})$ defines a functor, Δ_β, identifying the category $R^\sim - mod$ of left $R^\sim$-modules with a full subcategory of the category $R^\sim - bimod$ of $R^\sim$-bimodules.

Suppose the functor $X \longmapsto X \otimes -$ is right exact for any $X \in Ob\mathcal{A}$. Then the functor Δ_β identifies $R^\sim - mod$ with a monoidal subcategory of $R^\sim - bimod$. And the symmetry β induces a symmetry on $R^\sim - mod$.

3.1. Noncommutative affine schemes in a monoidal category. Fix a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1}, a)$. We define the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of *affine schemes in* $\mathcal{A}^\sim$ as the opposite category to the category of algebras: $\mathbf{Aff}_{\mathcal{A}^\sim} := (Alg\mathcal{A}^\sim)^{op}$. Thus if $\mathcal{A}^\sim = (k - mod, \otimes_k, k)$ for some commutative ring k, then $\mathbf{Aff}_{\mathcal{A}^\sim}$ is the category opposite to the category Alg_k of k-algebras.

We denote by $\mathbf{Spec}(R^\sim)$, or $\mathbf{Spec}_{\mathcal{A}^\sim}(R^\sim)$ the object of $\mathbf{Aff}_{\mathcal{A}^\sim}$ corresponding to the algebra $R^\sim$.

3.1.1. Monoidal functors and affine schemes. Recall that a monoidal functor from a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1}, a, l, r)$ to a monoidal category $\mathcal{A}'^\sim = (\mathcal{A}', \otimes', \mathbf{1}', a', l', r')$ is a triple $\Phi^\sim = (\Phi, \phi, \phi_0)$, where Φ is a functor $\mathcal{A} \longrightarrow \mathcal{A}'$, ϕ is a functor morphism $\{\phi_{X,Y} : \Phi(X) \otimes' \Phi(Y) \longrightarrow \Phi(X \otimes Y)$, ϕ_0 a morphism $\mathbf{1}' \longrightarrow \Phi(\mathbf{1}$ satisfying natural compatibility conditions.

Any monoidal functor $\Phi^\sim : \mathcal{A}^\sim \longrightarrow \mathcal{A}'$ induces a functor

$$\Phi^\sim_{Alg} : Alg\mathcal{A}^\sim \longrightarrow Alg\mathcal{A}'^\sim, \quad (R, m) \longmapsto \Phi^\sim(R, m) := (\Phi(R), \Phi(m) \circ \phi_{R,R}), \qquad (1)$$

hence a functor

$$\Phi^{\sim}_{\mathbf{Aff}} : \mathbf{Aff}_{\mathcal{A}^{\sim}} \longrightarrow \mathbf{Aff}_{\mathcal{A}'^{\sim}}$$

3.2. Coproducts and fiber products of affine schemes. Fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1}, a)$.

3.2.1. Lemma. *Suppose that the category $\mathcal{A}$ has products of $|J|$ objects. Then the category $Alg\mathcal{A}^{\sim}$ has products of $|J|$ objects.*

Proof. Let $R_i^{\sim} = (R_i, \mu_i)$, $i \in J$, be a set of algebras in $\mathcal{A}^{\sim}$. By assumption, there exists a product $\prod_{i\in J} R_i$ of the set of objects $\{R_i|\ i \in J\}$, and the canonical projections $p_j : \prod_{i\in J} R_i \longrightarrow R_j$ together with multiplications μ_j provide morphisms

$$\prod_{i\in J} R_i \otimes \prod_{i\in J} R_i \overset{p_j\otimes p_j}{\longrightarrow} R_j \otimes R_j \overset{\mu_j}{\longrightarrow} R_j, \quad j \in J. \tag{1}$$

By the universal property of products, there exists a unique morphism $\mu : \prod_{i\in J} R_i \otimes \prod_{i\in J} R_i \longrightarrow \prod_{i\in J} R_i$ such that for any $i \in J$, $p_i \circ \mu = \mu_i \circ p_i \otimes p_i$. We leave to the reader to check that thus defined morphism μ is a structure of an associative unital algebra on $\prod_{i\in J} R_i$ and that the algebra $(\prod_{i\in J} R_i, \mu)$ is a product of the set of algebras $\{R_i^{\sim} = (R_i, \mu_i),\ i \in J\}$. ∎

3.2.2. Proposition. *Suppose $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1}, a)$ is a monoidal category such that $\mathcal{A}$ has countable coproducts. Then*

For any algebras $R^{\sim} = (R, \mu)$ and $S^{\sim} = (S, \nu)$ in $\mathcal{A}^{\sim}$ such that the functors $R \otimes -$ and $S \otimes -$ are compatible with countable coproducts, there exists a free product $R^{\sim} \star S^{\sim}$ which is a coproduct in the category $Alg\mathcal{A}^{\sim}$. In particular, $\mathbf{Spec}_{\mathcal{A}^{\sim}}(R^{\sim} \star S^{\sim})$ is a product of $\mathbf{Spec}_{\mathcal{A}^{\sim}}(R^{\sim})$ and $\mathbf{Spec}_{\mathcal{A}^{\sim}}(S^{\sim})$ in the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$.

Proof. Let $\mathfrak{L}^{\sim}_{\mathcal{A}^{\sim}} = (\mathfrak{L}, a)$ be the canonical monoidal functor "of left multiplication"

$$\mathcal{A}^{\sim} \longrightarrow End^{\sim}\mathcal{A},\ M \longmapsto M \otimes -$$

The monoidal functor $\mathfrak{L}^{\sim}_{\mathcal{A}^{\sim}}$ induces a functor from $Alg\mathcal{A}^{\sim}$ to $\mathfrak{Mon}\mathcal{A}$ which assigns to algebras $R^{\sim} = (R, \mu)$ and $S^{\sim} = (S, \nu)$ the monads $(\mathfrak{L}(R), \mu_R)$ and $(\mathfrak{L}(S), \nu_S)$ such that the functors $\mathfrak{L}(R)$ and $\mathfrak{L}(S)$ preserve countable coproducts. The assertion follows now from Proposition 2.6.2.3. ∎

3.2.3. Proposition. *Let $\mathcal{A}$ be a category with countable colimits. And suppose that for any $M \in Ob\mathcal{A}$, the functor $M \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$ is compatible with countable colimits. Then the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$ has fiber products.*

Proof. Let $R^{\sim} \longleftarrow S^{\sim} \longrightarrow T^{\sim}$ be morphisms of $Alg\mathcal{A}^{\sim}$. Since the functor $M \longmapsto M \otimes -$ preserves cokernels of coreflexive pairs of arrows, we have well defined monads $\mathbb{F} = R^{\sim} \otimes_{S^{\sim}} -$ and $\mathbb{G} = T^{\sim} \otimes_{S^{\sim}} -$ on the category $S^{\sim} - mod$ of left $S^{\sim}$-modules.

The fact that the functors $R \otimes -$ and $T \otimes -$ preserve countable coproducts implies that the functors $R \otimes_{S^{\sim}} -$ and $T \otimes_{S^{\sim}} -$ from $S^{\sim} - mod$ to $S^{\sim} - mod$ have the same property.

By 2.6.2.3, there exists the coproduct $\mathbb{F}\star\mathbb{G}$ which we denote by (F',μ'). To the monad (F',μ') on $S^\sim - mod$, there corresponds a monad $(f_*F'f^*,\mu'_f)$ on $\mathcal{A}$. Here f_* denotes the canonical forgetful functor $S^\sim - mod \longrightarrow \mathcal{A}$ and f^* is its left adjoint: $L \longmapsto (S\otimes L, m_L)$, where m_L is the canonical action induced by the multiplication on S. The multiplication μ'_f is the composition of the canonical morphism $f_*F'f^*f_*F'f^* \longrightarrow f_*F'^2f^*$ induced by the adjunction morphism $f^*f_* \longrightarrow Id$ and the morphism $f_*\mu'f^* : f_*F'^2f^* \longrightarrow f_*F'f^*$. With the monad $(f_*F'f^*,\mu'_f)$ one can associate the algebra $(f_*F'f^*(\mathbf{1}),\mu'') = (f_*F'(S),\mu'')$, where μ'' is a naturally defined multiplication. This algebra is a fiber coproduct of $R^\sim$ and $T^\sim$ over $S^\sim$. A less formal (more constructive) argument uses the explicit description of $f_*F'(S)$ in terms of tensor product of copies of $R^\sim$ and $S^\sim$ over $S^\sim$ (following 2.6.2) in which the multiplication appears immediately and the universal property is evident. We leave this argument to the reader. ■

3.2.4. Example. Let $\mathcal{A}^\sim = (k - mod^\sim) = (k - mod, \otimes_k, k)$ for a commutative associative ring k. Algebras in $k - mod^\sim$ are just k-algebras, and the coproduct (or $\star$-product) of algebras A and B is their free product. For instance, the coproduct of two copies of polynomial algebra in one variable is isomorphic to the free algebra in two variables: $k[x]\star k[y] \simeq k\langle x,y\rangle$.

3.3. A 'base space' and quasi-coherent modules. A link between the formal notion of an affine scheme in a monoidal category and a more geometric notion of an affine scheme over a category C is given by fixing an action of the monoidal category on the category C, i.e. a triple $(\mathcal{A}^\sim, C, \Phi^\sim)$, where C is a category, $\mathcal{A}^\sim = (\mathcal{A},\otimes,\mathbf{1},a)$ is a monoidal category, $\Phi^\sim = (\Phi,\phi)$ is a monoidal functor $\mathcal{A}^\sim \longrightarrow End_2^\sim C$. Recall that End_2C denotes the full subcategory of the category $EndC$ objects of which are functors compatible with cokernels of reflexive pairs of morphisms (cf. 1.2.2). In other words, $\Phi^\sim$ is a unital action of the monoidal category $\mathcal{A}^\sim$ on a category C which preserves cokernels of reflexive pairs of arrows.

We shall call the triple $(\mathcal{A}^\sim, C, \Phi^\sim)$ a *'base space'*, or simply a *base*.

3.3.1. Quasi-coherent modules on an affine scheme. The action $\Phi^\sim$ induces a functor, $R^\sim \longmapsto \Phi^\sim(R^\sim)$, from the category $Alg\mathcal{A}^\sim$ to the category of monads on the category C (cf. 3.1.1(1)). In particular, we have the category $\Phi^\sim(R^\sim) - mod_C$ of $\Phi^\sim(R^\sim)$-modules which we call simply $R^\sim$-modules in C, or quasi-coherent $\mathcal{O}_{\mathbf{Spec}(R^\sim)}$-modules over $\mathfrak{C}$, and denote by $Qcoh_{\mathbf{Spec}(R^\sim)/\mathfrak{C}}$.

3.3.2. A fibered category associated with a base. We associate with $\mathfrak{C}$ a cofibered category $\mathcal{F}^{\mathfrak{C}} \stackrel{\pi}{\longrightarrow} (Alg\mathcal{A}^\sim)^{op} = \mathbf{Aff}_{\mathcal{A}^\sim}$ in a natural way: objects of the category $\mathcal{F}^{\mathfrak{C}}$ are pairs $(R^\sim, M)$, where $R^\sim = (R,\mu)$ is an algebra in $\mathcal{A}^\sim$, M is a $\Phi^\sim(R^\sim)$-module. A morphism from $(R^\sim, M)$ to $(S^\sim, L)$ is a pair (ψ,ξ), where ψ is an algebra morphism $R^\sim \longrightarrow S^\sim$, ξ is a morphism $M \longrightarrow \psi_*(L)$, where ψ_* is the pull-back functor $\Phi^\sim(R^\sim) - mod_C \longrightarrow \Phi^\sim(S^\sim) - mod_C$ induced by the algebra morphism ψ. The composition is defined in a standard way. The projection π assigns to any object $(R^\sim, M)$ the algebra $R^\sim$ and to any morphism (ψ,ξ) the algebra morphism ψ. It follows from the assumptions and 1.0.6.1 that $\mathcal{F}^{\mathfrak{C}} \stackrel{\pi}{\longrightarrow} \mathbf{Aff}_{\mathcal{A}^\sim}$ is a bifibered category: the functors ψ_* have left adjoints.

3.4. Examples.

3.4.1. A typical noncommutative example is the monoidal category $R - bimod^{\sim} = (R - bimod, \otimes_R, R)$ of bimodules over an associative ring R acting (by tensoring over R) on the category of left R-modules. For any algebra S in the monoidal category $R - bimod^{\sim}$ (that is a ring morphism $R \longrightarrow S$), the category of quasi-coherent modules on $\mathbf{Spec}(S)/\mathfrak{C}$ is naturally equivalent to the category $S - mod$ of left S-modules.

3.4.2. The left and right 'base spaces' of a monoidal category. Let $\mathcal{A}^{\sim} = (\mathcal{A}, \otimes, \mathbf{1})$ be a monoidal category such that for any $X \in Ob\mathcal{A}$, the functor $X \otimes - : V \longmapsto X \otimes V$ preserves cokernels of reflexive pairs of morphisms. The *'left base space' of the monoidal category* $\mathcal{A}^{\sim}$ is the triple $(\mathcal{A}, \mathcal{A}^{\sim}, (\mathfrak{L}, a))$, where $\mathfrak{L} = \mathfrak{L}_{\mathcal{A}^{\sim}}$ is the functor $\mathcal{A} \longrightarrow End\mathcal{A}$, $X \longmapsto X \otimes -$, a is the associativity constraint. The functor $\Phi^{\sim}$ is faithful, because $X \longmapsto X \otimes \mathbf{1}$ is a faithful functor.

Replacing the action $X \longmapsto X \otimes -$ by $X \longmapsto - \otimes X$, we obtain the notion of the *right base space* of the monoidal category $\mathcal{A}^{\sim}$.

3.4.3. Operads and algebras over operads. Fix a symmetric additive monoidal category $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, \mathbf{1}, a, l, r; \beta)$ (here β is a symmetry, $\beta_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$). Let $\mathbf{S}$ denote the category objects of which are sets $[n] := \{1, ..., n\}$, $n \geq 1$, and $[0] := \emptyset$ and morphisms are bijections. Denote by $\mathcal{C}^{\mathbf{S}}$ the category of functors $\mathbf{S}^{op} \longrightarrow \mathcal{C}$. In other words, objects of $\mathcal{C}^{\mathbf{S}}$ are collections $\mathcal{M} = (M(n)|\ n \geq 0)$, where M_n is an object of $\mathcal{C}$ with an action of the symmetric group S_n.

The category $\mathcal{C}^{\mathbf{S}}$ acts on the category $\mathcal{C}$ by *polynomial functors*:

$$M : V \longmapsto M(V) = \oplus_{n \geq 0} M(n) \otimes_{S_n} V^{\otimes n} \tag{1}$$

The composition of polynomial functors is again a polynomial functor. This defines a tensor product, $\odot$, on $\mathcal{C}^{\mathbf{S}}$ called the *plethism product*. We denote the corresponding monoidal category $(\mathcal{C}^{\mathbf{S}}, \odot, \mathbf{1_S})$ by $\mathcal{C}^{\sim\mathbf{S}}$. Here $\mathbf{1_S}$ is the unit object $\mathbf{1_S}$. One can see that $\mathbf{1_S}(n) = 0$ if $n \neq 1$ and $\mathbf{1_S}(1)$ is the unit object of the category $\mathcal{C}^{\sim}$. Thus we have an action $\mathfrak{C}$ of the monoidal category $\mathcal{C}^{\mathbf{S}}$ on the category $\mathcal{C}$.

Algebras in the monoidal category $\mathcal{C}^{\mathbf{S}}$ are called *operads* or $\mathcal{C}^{\sim}$*-operads*. For each operad $\mathcal{R}$, the corresponding category of quasi-coherent sheaves on $\mathbf{Spec}(\mathcal{R})$ is usually called *the category of* $\mathcal{R}$*-algebras*.

3.5. Affine schemes and relative affine schemes. Fix a base $\mathfrak{C} = (\mathcal{A}^{\sim}, C, \Phi^{\sim})$. Let $\mathbf{Spec}_{\mathfrak{C}}$ denote the functor $(Alg\mathcal{A}^{\sim})^{op} = \mathbf{Aff}_{\mathcal{A}^{\sim}} \longrightarrow RCat/C$ which assigns to any affine scheme $\mathbf{X} = \mathbf{Spec}(R^{\sim})$ in $\mathcal{A}^{\sim}$ the object $(\Phi^{\sim}(R^{\sim}) - mod_C, f_C) := (Qcoh_{\mathbf{X}/\mathfrak{C}}, f_C)$, where f_C is the canonical ($\mathfrak{S}_2$-affine) morphism $\Phi^{\sim}(R^{\sim}) - mod_C \longrightarrow C$, and to any affine morphism $\mathbf{Spec}(R^{\sim}) \longrightarrow \mathbf{Spec}(S^{\sim})$ the ($\mathfrak{S}_2$-affine) morphism direct image of which is the corresponding pull-back functor. There is a canonical morphism from the fibered category $\mathcal{F}^{\mathfrak{C}} \xrightarrow{\pi} (Alg\mathcal{A}^{\sim})^{op}$ to the fibered category $RCat_2^{\sim C}$ which assigns to any object $(R^{\sim}, M)$ of $\mathcal{F}^{\mathfrak{C}}$ the object $(\mathbf{Spec}_{\mathfrak{C}}(R^{\sim}), M)$.

3.5.1. $\mathfrak{C}$-affine morphisms. Morphisms of the form $\mathbf{Spec}_{\mathfrak{C}}(\psi)$ will be called $\mathfrak{C}$-*affine*, or, loosely, *affine*. They are not usually affine in the sense of 2.3, as one can see taking as a base the monoidal category $End_2^{\sim}C$ and as $\Phi^{\sim}$ the identical monoidal functor.

Let $\mathfrak{C}$ be the 'left base space' $(\mathcal{A}^\sim, \mathfrak{L}^\sim)$ of a monoidal category $\mathcal{A}^\sim$ satisfying the conditions of 3.4.2.

3.5.2. Lemma. *Let $\mathcal{R} = (R, \mu)$ and $\mathcal{S} = (S, \nu)$ be algebras in $\mathcal{A}^\sim$ and ψ an algebra morphism $\mathcal{S} \longrightarrow \mathcal{R}$. The morphism $\mathbf{Spec}_{\mathfrak{C}}(\psi) : \mathcal{R} - mod \longrightarrow \mathcal{S} - mod$ is affine iff the inner hom $\mathbf{Hom}_{\mathcal{S}}(R, M)$ exists for all left $\mathcal{S}$-modules M.*

Proof. Suppose the inner hom $\mathbf{Hom}_{\mathcal{S}}(R, M)$ exists for all left $\mathcal{S}$-modules M. It has a canonical left action of $\mathcal{R}$. For any left $\mathcal{R}$-module L, we have a canonical functorial isomorphism

$$Hom_{\mathcal{R}}(L, \mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}), M) \xrightarrow{\sim} Hom_{\mathcal{S}}(\psi_*(\mathcal{R} \otimes_{\mathcal{R}} L), M) = Hom_{\mathcal{S}}(\psi_*(L), M) \qquad (1)$$

which shows that the functor $M \longmapsto \mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}, M)$ is a right adjoint to the direct image functor ψ_*. The isomorphism (1) can be regarded as a definition of $\mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}, M)$. In particular, the existence of a right adjoint to ψ_* implies that of $\mathbf{Hom}_{\mathcal{S}}(\psi_*(\mathcal{R}), \mathbf{M})$. ■

3.5.2.1. Note. In the case $\mathcal{A}^\sim = k - mod^\sim := (k - mod, \otimes_k, k)$ for a commutative ring k, algebras in $\mathcal{A}^\sim$ are k-algebras and, given an algebra morphism $\mathcal{S} \longrightarrow \mathcal{R} = (R, \mu)$ and an $\mathcal{S}$-module M, the inner hom, $\mathbf{Hom}_{\mathcal{S}}(R, M)$, coincides with $Hom_{\mathcal{S}}(R, M)$.

3.6. The base change. A morphism from a base $\mathfrak{C}' = (\mathcal{A}'^\sim, C', \Phi'^\sim)$ to a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ is a triple $(\Psi^\sim, g^*; \lambda)$, where $\Psi^\sim$ is a monoidal functor $\mathcal{A}'^\sim \longrightarrow \mathcal{A}^\sim$ and g^* an inverse image functor of a morphism $C' \longrightarrow C$, and λ a functorial isomorphism $C(M, \Phi\Psi(E')(L)) \xrightarrow{\sim} C'(g^*(M), \Phi'(E')(g^*(L)))$. We leave to the reader defining the composition. The main example is as follows.

3.6.1. Fix a 'base space' $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ such that the category C has cokernels of reflexive pairs of arrows. Let $R^\sim = (R, \mu)$ be an algebra in the monoidal category $\mathcal{A}^\sim$. We assume that for any $M \in Ob\mathcal{A}$, the functors $M \otimes -$ and $- \otimes M$ preserve cokernels of reflexive pairs of arrows. In this case we have a well defined monoidal category $R^\sim - bimod^\sim = (R^\sim - bimod, \otimes_{R^\sim} -, R^\sim)$. Moreover, there is a naturally defined action of the monoidal category $R^\sim - bimod^\sim$ on the category $\Phi^\sim(R^\sim) - mod_C$.

In fact, for any $R^\sim$-bimodule $\mathcal{M} = (m', M, m'')$ and for any $\Phi^\sim(R^\sim)$-module (L, ν), the action of $\mathcal{M}$ on (L, ν) assigns to (L, ν) the $\Phi^\sim(R^\sim)$-module $\Phi^\sim(\mathcal{M}) \otimes_{\Phi^\sim(R^\sim)} L$. The latter is the cokernel of the pair of arrows $\Phi^\sim(\mathcal{M})\Phi^\sim(R^\sim)(L) \rightrightarrows \Phi^\sim(\mathcal{M})(L)$, where one arrow is the composition of the constraint

$$\phi_{\mathcal{M},R}(L) : \Phi^\sim(\mathcal{M})\Phi^\sim(R^\sim)(L) \longrightarrow \Phi^\sim(\mathcal{M} \otimes R^\sim)(L)$$

and the morphism $\Phi^\sim(m'')(L)$ induced by the right action of $R^\sim$ on $\mathcal{M}$. The other arrow is the image $\Phi^\sim(\mathcal{M})(\nu)$ of the left action on L.

We have a morphism $g : R^\sim - mod \longrightarrow C = \mathcal{A}$ with the inverse image functor $g^* : L \longmapsto R^\sim \otimes L$ and the functor $\Psi : \mathcal{A} \longrightarrow R^\sim - bimod$, $M \longmapsto R^\sim \otimes M \otimes R^\sim$. The latter extends canonically to a monoidal functor $\Psi^\sim : \mathcal{A}^\sim \longrightarrow R^\sim - bimod^\sim$.

3.7. The fibered category of bimodules. Fix a monoidal category $\mathcal{A}^\sim$ such that for all $M \in Ob\mathcal{A}$, the functors $M \otimes -$ preserve colimits of reflexive pairs of arrows. We associate with $\mathcal{A}^\sim$ a cofibered category $Bi^{\mathcal{A}^\sim} \xrightarrow{\pi} (Alg\mathcal{A}^\sim)^{op} = \mathbf{Aff}_{\mathcal{A}^\sim}$ defined as follows. Objects of the category $Bi^{\mathcal{A}^\sim}$ are pairs $(R^\sim, M)$, where $R^\sim = (R, \mu)$ is an algebra in $\mathcal{A}^\sim$, M is an $R^\sim$-bimodule. A morphism from $(R^\sim, M)$ to $(S^\sim, L)$ is a pair (ψ, ξ), where ψ is an algebra morphism $R^\sim \longrightarrow S^\sim$, ξ is an $S^\sim$-bimodule morphism $M \longrightarrow \psi_*(L)$, where ψ_* is the pull-back functor $S^\sim - bimod \longrightarrow R^\sim - bimod$ induced by the algebra morphism ψ. The composition is defined in a standard way. The projection π assigns to any object $(R^\sim, M)$ the algebra $R^\sim$ and to any morphism (ψ, ξ) the algebra morphism ψ. It follows from the assumptions and 1.0.6.1 that $Bi^{\mathcal{A}^\sim} \xrightarrow{\pi} \mathbf{Aff}_{\mathcal{A}^\sim}$ is a bifibered category: the functors ψ_* have left adjoints. For any $R^\sim \in ObAlg\mathcal{A}^\sim$, the fiber $R^\sim - bimod$ over $\mathbf{Spec}(R^\sim)$ will be regarded as a monoidal category with respect to $\otimes_{R^\sim}$. For any algebra morphism $\psi : R^\sim \longrightarrow S^\sim$, the corresponding direct image functor ψ_* is a monoidal functor (in a week sense). This monoidal structure is inherited by the fibered category $\mathfrak{Qcoh}(Bi^{\mathcal{A}^\sim})$ of quasi-coherent morphisms of $Bi^{\mathcal{A}^\sim}$ (cf. 1.0.1) and, therefore, by the category $Qcoh(Bi^{\mathcal{A}^\sim})$ of quasi-coherent presheaves of bimodules (see 1.0.2).

Fix a 'base space' $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ such that the category C has cokernels of reflexive pairs of arrows. It follows from 3.6 that the action $\Phi^\sim$ of the monoidal category $\mathcal{A}^\sim$ on the category C induces an action of the fibered category $Bi^{\mathcal{A}^\sim}$ on the fibered category $\mathcal{F}^{\mathfrak{C}}$ defined in 3.4. This action induces an action of $\mathfrak{Qcoh}(Bi^{\mathcal{A}^\sim})$ on the corresponding fibered category $\mathfrak{Qcoh}(\mathcal{F}^{\mathfrak{C}})$ of quasi-coherent morphisms.

3.8. Fibered categories and quasi-coherent presheaves associated with a functor from $\mathbf{Aff}_{\mathcal{A}^\sim}$. Let G be a functor $\mathbf{Aff}_{\mathcal{A}^\sim} := (Alg\mathcal{A}^\sim)^{op} \longrightarrow \mathcal{S}$. For any $X \in Ob\mathcal{S}$, we have a natural functor $G/X \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$ and the corresponding pull-back, $\mathcal{F}^{\mathfrak{C},G/X}$ (see 1.0.4), of the fibered category $\mathcal{F}^{\mathfrak{C}}$ (cf. 3.4). In particular, we have the category $PQcoh(\mathcal{F}^{\mathfrak{C},G/X})$ of quasi-coherent presheaves on X (cf. 1.0.4). Since the construction is functorial in X, it gives a rise to a fibered category over $\mathcal{S}$ having $PQcoh(\mathcal{F}^{\mathfrak{C},G/X})$ as a fiber at an object X.

Similarly, one can define the monoidal *category of quasi-coherent bimodules* on each object X of the category S as the category of quasi-coherent presheaves of the fibered category $\pi_{G/X} : Bi^{\mathcal{A}^\sim, G/X} \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$ obtained as the pull-back by the canonical functor $G/X \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$.

3.8.1. Quasi-coherent presheaves on a presheaf of sets. The case of a particular interest is when the functor G is the canonical embedding

$$h : \mathbf{Aff}_{\mathcal{A}^\sim} \longrightarrow Fun(\mathbf{Aff}_{\mathcal{A}^\sim}^{op}, \mathbf{Sets}), \quad X \longmapsto \mathbf{Aff}_{\mathcal{A}^\sim}(-, X)$$

of the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ to the category of presheaves of sets on $\mathbf{Aff}_{\mathcal{A}^\sim}$. We have then the notion of the category $PQcoh_X$ of quasi-coherent presheaves on a presheaf of sets X and the monoidal category $PQcoh(Bi_X)$ of quasi-coherent bimodules on X together with an action of $PQcoh(Bi_X)$ on $PQcoh_X$.

4. Locally affine $\mathfrak{T}$-spaces.

The contents of this section is a version of a formalism of glueing schemes and algebraic spaces from affine schemes [GrD], [A1], [Kn]. It is convenient for us to develop this formalism in a general setting, starting from a version of a Grothendieck topology (like in the first chapter of [Kn]).

The known examples of noncommutative locally affine spaces appear either as categories (of quasi-coherent sheaves) over a base category having affine covers, like, for instance, the flag variety of a quantized enveloping algebra (see [R3]), or as sheaves of sets on the category of noncommutative affine schemes, like projective spaces and Grassmannians introduced in [KR]. Respectively, we introduce the notions of a scheme and that of a locally affine space (noncommutative projective spaces of [KR] are locally affine spaces, but not schemes) in two different settings. Their common feature is as follows. We start from a category A with a subcanonical (quasi-)topology $\mathfrak{T}$ on A and a functor $F : A \longrightarrow B$. In terms of this data, we define the category of 'locally affine' objects. In our applications the category A is the category of $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes in a monoidal category $\mathcal{A}^\sim$ and B is a bigger category of 'spaces'. The two basic settings are as follows:

(a) B is the category of sheaves on $(A, \mathfrak{T})$ and F is the canonical embedding.

(b) $B = RCat/C$ for some category C and the functor F is determined by an action of the monoidal category $\mathcal{A}^\sim$ on C.

The model commutative example of this setting is the site of affine schemes with étale or Zariski topology, the functor F being the embedding of the category of affine schemes into the category of locally ringed spaces. In this case the definition of $(\mathfrak{T}|F)$-schemes coincides with the standard definition (i.e. the one via affine covers) of schemes. If $\mathfrak{T}$ is the étale topology, locally affine $(\mathfrak{T}|F)$-spaces are algebraic spaces in the sense of Artin.

The main noncommutative setting is the site of affine $\mathcal{A}^\sim$-schemes with the **fpqc** topology corresponding to a fixed base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ and the embedding of the category of noncommutative affine schemes into the category $RCat/C$ of 'spaces' over C.

Section 4.1 contains preliminaries on sites.

In Sections 4.2, 4.3 we introduce a version of the **fpqc** topology and that of Zariski topology on the category of noncommutative affine schemes depending on a fixed base (i.e. on an action of a monoidal category on a category) and prove certain descent property of these topologies.

In Section 4.5 we define *locally affine spaces* and *schemes* on a subcanonical quasi-site $(A, \mathfrak{T})$ (i.e. the topology $\mathfrak{T}$ is subcanonical) called respectively locally affine $\mathfrak{T}$-spaces and $\mathfrak{T}$-schemes. Informally speaking, these are sheaves of sets on the site $(A, \mathfrak{T})$ glued of representable sheaves.

In Sections 4.5 and 4.6, given a site $(A, \mathfrak{T})$ and a functor $F : A \longrightarrow B$, we introduce the category of $(\mathfrak{T}|F)$-spaces and its full subcategories formed resp. by *locally affine* $(\mathfrak{T}|F)$-*spaces* and by $(\mathfrak{T}|F)$-*schemes*.

In section 4.6 we establish a connection between (locally affine) $(\mathfrak{T}|F)$-spaces and $(\mathfrak{T}|F)$-schemes and (locally affine) $\mathfrak{T}$-spaces and $\mathfrak{T}$-schemes. This connection is given by a natural functor from the category of $(\mathfrak{T}|F)$-spaces to the category of $\mathfrak{T}$-spaces which are by definition sheaves of sets on the site $(A, \mathfrak{T})$.

4.1. Preliminaries on sites. Let A be a category. A *sieve* of A is a subcategory R of A such that for any $X \in ObR$ all morphisms $Y \longrightarrow X$ belong to R (in particular, R is a full subcategory of A). If $F : A' \longrightarrow A$ is a functor and R a sieve of A, then $R^F := F^{-1}(R)$ is a sieve of A' called *inverse image of* R *by* F.

An arrow $f : X \longrightarrow Y$ in A induces a functor $A/f : A/X \longrightarrow A/Y$. If R is a sieve in A/Y, let R^f denote its inverse image by A/f. For a family of arrows $\mathbf{U} = (U_i \longrightarrow U|\ i \in J)$, denote by $R_{\mathbf{U}}$ the full subcategory of R/U objects of which are those $X \longrightarrow U$ that there exists a morphism $(X \longrightarrow U) \longrightarrow (U_i \longrightarrow U)$ for some $i \in J$. This is a sieve called *the sieve of A/U generated by* $\mathbf{U}$.

If $f : V \longrightarrow U$ and there exist $U_i \times_U V$, the inverse image $R^f_{\mathbf{U}}$ of $R_{\mathbf{U}}$ by f is the sieve generated by projections $(U_i \times_U V \longrightarrow V|\ i \in J)$.

4.1.1. Definition of a quasi-topology. A *quasi-topology* τ on a category A is a function which assigns to any object X of the category A a set $Cov_\tau(X)$ of families of arrows $\mathbf{U} = \{U_i \longrightarrow U|\ i \in J\}$ such that

(a) For any $X \in ObA$, $(id_X) \in Cov_\tau(X)$.

(b) If $\{u_i : U_i \longrightarrow X|i \in J\} \in Cov_\tau(X)$ and $\{u_{ij} : U_{ij} \longrightarrow U_i|j \in J_i\} \in Cov_\tau(U_i)$ for any $i \in J$, then $\{u_i \circ u_{ij} : U_{ij} \longrightarrow X|i \in J,\ j \in J_i\} \in Cov_\tau(X)$.

The cover $\{u_i \circ u_{ij} : U_{ij} \longrightarrow X|i \in J,\ j \in J_i\}$ is called a refinement of the cover $\{u_i : U_i \longrightarrow X|i \in J\}$.

(c) Any two covers of an object have a common refinement.

4.1.2. Continuous morphisms and continuous quasi-topologies. Given a quasi-topology τ on the category A, we call a morphism $f : X \longrightarrow Y$ *continuous*, or *τ-continuous*, if for any $\mathbf{U} \in Cov_\tau(Y)$, the sieve $R^f_{\mathbf{U}}$ of A/X (cf. 4.1.0) contains an element of $Cov_\tau(X)$.

We call a quasi-topology τ *continuous* if any morphism of A is continuous.

4.1.2.1. The subcategory A^τ. Fix a quasi-topology τ on A.

Denote by A^τ the class of all morphisms of A which belong to some τ-cover. It follows from 4.1.1(a) and 4.1.1(b) that A^τ is a subcategory of A. Clearly τ induces a quasi-topology on A^τ. The property 4.1.1(c) implies that this quasi-topology is continuous.

4.1.2.2. The subcategory A_τ. Denote by A_τ the class of all morphisms $f : X \longrightarrow Y$ of A such that for any τ-cover $\mathbf{U}$ of Y, the sieve $R^f_{\mathbf{U}}$ (cf. 4.1) contains a τ-cover of X. Clearly the composition of morphisms of A_τ belongs to A_τ as well as all identical morphisms of the category A. This shows that A_τ is a subcategory of A and $ObA_\tau = ObA$. The subcategory A_τ contains the subcategory A^τ (see 4.1.2.1), hence the quasi-topology τ induces a continuous quasi-topology on A_τ. Clearly, A_τ is a maximal subcategory of A having this property. In particular, if $A_\tau = A$ iff τ is continuous.

4.1.3. Continuous quasi-topologies and topologies.

4.1.3.1. Recall that a *topology* $\mathfrak{T}$ on a category A is a map which assigns to each $X \in ObA$ a non-empty set $\mathfrak{T}(X)$ of sieves of the category A/X such that

(a) For any morphism $f : Y \longrightarrow X$ of A and any $R \in \mathfrak{T}(X)$, $R^f \in \mathfrak{T}(Y)$.

(b) For any $S \in ObA$ and any $R \in \mathfrak{T}(S)$, any sieve R_1 of A/S such that $R^f_1 \in \mathfrak{T}(Y)$ for all $(Y \xrightarrow{f} S) \in R$ belongs to $\mathfrak{T}(S)$.

For any $X \in ObA$, the elements of $\mathfrak{T}(X)$ are called *refinements* of X. The pair $(A, \mathfrak{T})$ is called *site*.

One checks easily that for any $X \in ObA$, the intersection of two refinements of X is a refinement of X and that any sieve of A/X which contains a refinement is a refinement. In particular A/X is a refinement.

4.1.3.2. Covers of a topology. A set of arrows $\mathbf{U} = \{U_i \longrightarrow U|\ i \in J\}$ is called a *cover* in the topology $\mathfrak{T}$ if the sieve $R_{\mathbf{U}}$ generated by $\mathbf{U}$ belongs to $\mathfrak{T}(U)$.

4.1.3.3. A topology associated with covers. Let τ be a function which assigns to any object X of the category A a set $Cov_\tau(X)$ of families of arrows $\mathbf{U} = \{U_i \longrightarrow U|\ i \in J\}$ such that for any $X \in ObA$, $(id_X) \in Cov_\tau(X)$. The *topology generated by* τ is the coarsest topology for which all $\mathbf{U} \in Cov_\tau(X)$, $X \in ObA$, are covers. We denote this topology by $\mathfrak{T}_\tau$ and call τ *a base of the topology* $\mathfrak{T}_\tau$.

4.1.3.4. Lemma. *Suppose that τ is a continuous quasi-topology. Then the topology $\mathfrak{T}_\tau$ generated by τ is described as follows. For any $X \in ObA$, $\mathfrak{T}_\tau(X)$ consists of all sieves R of A/X containing some element of $Cov_\tau(X)$.*

Proof. Denote by $\mathfrak{T}$ the function which assigns to any $X \in ObA$ the set of sieves R of A/X such that R contains some $\mathbf{U} \in Cov_\tau(X)$. Obviously $\mathfrak{T}(X) \subset \mathfrak{T}_\tau(X)$ for all $X \in ObA$. To prove the inverse inclusion it suffices to show that the function $\mathfrak{T}$ is a topology, i.e. it satisfies the conditions (a) and (b) of the definition of topology (cf. 4.1.3.1).

The condition (a) in follows from the fact that all morphisms of A are τ-continuous. It remains to check the condition (b) of 4.1.3.1.

Let $R \in \mathfrak{T}(X)$, i.e. R is a sieve of A/X containing some $\mathbf{U} \in Cov_\tau(X)$. And let R_1 be a sieve of A/X such that $R_1^f \in \mathfrak{T}$ for any $f \in R$. In particular for any $(U_f \xrightarrow{f} X) \in \mathbf{U}$, the sieve R_1^f contains some $\mathbf{U}^f \in Cov_\tau(U_f)$. This implies (by definition of R_1^f) that R_1 contains the composition of $\mathbf{U}$ and $\{\mathbf{U}^f|\ f \in \mathbf{U}\}$. Since this composition belongs to $Cov_\tau(X)$ by the definition of a base of a topology, $R_1 \in \mathfrak{T}(X)$. Thus $\mathfrak{T}$ is a topology, hence the assertion. ■

4.1.4. Pretoplogies. Recall that a quasi-topology τ is called a *pretopology* if the following condition holds:

(c) For any morphism $V \longrightarrow X$ and cover $\mathfrak{U} = \{u_i : U_i \longrightarrow X|i \in J\} \in Cov_\tau(X)$, there exist fiber products $U_i \times_X V$ for all $i \in J$, and the set of canonical projections $\mathfrak{U} \times_X V := \{p_i : U_i \times_X V \longrightarrow V|i \in J\}$ belongs to $Cov_\tau(V)$.

Clearly any pretopology is continuous.

4.1.5. Presheaves and sheaves. A presheaf of sets on a category B is any functor $B^{op} \longrightarrow$ **Sets**. A presheaf of sets on a quasi-site (A, τ) is a presheaf on A.

A presheaf of sets F on a quasi-site (A, τ) is called a *sheaf* if for any $X \in ObA$ and any $\mathbf{U} \in Cov_\tau$, the natural morphism $F(X) \longrightarrow lim(F|R_{\mathbf{U}})$ is an isomorphism. Here $F|R_{\mathbf{U}}$ is the presheaf of sets on the sieve $R_{\mathbf{U}}$ generated by $\mathbf{U}$ which assigns to any object $Y \longrightarrow X$ of $R_{\mathbf{U}}$ the set $F(Y)$ and to any morphism $(Y \longrightarrow X) \xrightarrow{\lambda} (Y' \longrightarrow X)$ of $R_{\mathbf{U}}$, the map $F(\lambda) : F(Y') \longrightarrow F(Y)$.

4.1.5.1. Proposition. *Let τ be a quasi-topology on the category A such that for any cover $\{U_i \longrightarrow U|\ i \in J\} \in \tau(U)$, all fiber products $U_i \times_U U_j$, $i, j \in J$, exist. A presheaf* *of sets F on the category A is a sheaf for the quasi-topology $\mathfrak{T}_\tau$ generated by τ iff for any $X \in ObA$ and for any cover $\{U_i \longrightarrow U|\ i \in J\}$, the diagram of sets*

$$F(U) \longrightarrow \prod_{i\in J} F(U_i) \rightrightarrows \prod_{i,j\in J} F(U_i \times_U U_j)$$

is exact.

4.1.5.2. Subcanonical quasi-topologies. Any quasi-topology $\mathfrak{T}$ on A such that all representable presheaves are sheaves on the quasi-site (A, τ), is called *subcanonical.*

4.2. Natural quasi-topologies on the category of affine schemes. Fix a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim : \mathcal{A}^\sim \longrightarrow End_2^\sim C)$.

4.2.1. The fp quasi-topology. We call a morphism $w : W \longrightarrow X$ of $\mathbf{Aff}_{\mathcal{A}^\sim}$ *flat* if $\mathbf{Spec}_{\mathfrak{C}}(w)$ preserves kernels of coreflexive pairs of arrows and *faithfully flat* and if in addition it reflects isomorphisms. Composition of faithfully flat morphisms is a faithfully flat morphism. If morphisms $X \longrightarrow Z \longleftarrow Y$ are faithfully flat, then the projections $X \longleftarrow X \times_Z Y \longrightarrow Y$ are faithfully flat. Hence faithfully flat morhisms regarded as covers form a quasi-topology $\tau^{\mathfrak{C}}_{\mathbf{fp}}$ which we call the *faithfully flat*, shortly **fp** *quasi-topology.* Unlike the commutative case, the **fp** quasi-topology is not a pretopology in general.

4.2.1.1. The fp quasi-topology in the case of the left base. Let $\mathfrak{C}$ be the left base of a monoidal category $\mathcal{A}^\sim$, i.e. $\mathfrak{C} = (\mathcal{A}^\sim, \mathcal{A}, (\mathfrak{L}, a))$ (cf. 3.4.2). Fix algebras $R^\sim$, $S^\sim$ in $\mathcal{A}^\sim$ and an algebra morphism $\psi : R^\sim \longrightarrow S^\sim$. A corresponding inverse image functor ψ^* (a left inverse to the pull-back functor ψ_*) is isomorphic to $S^\sim \otimes_{R^\sim} -$. The composition $\psi^*\psi_* : S^\sim - mod_{\mathcal{A}} \longrightarrow S^\sim - mod_{\mathcal{A}}$ is isomorphic to the functor $(S^\sim \otimes_{R^\sim} S^\sim) \otimes_{S^\sim} -$. The functor $\psi^*\psi_*$ has a canonical structure of a comonad $\delta_\psi := \psi^*\eta_\psi\psi_* : \psi^*\psi_* \longrightarrow (\psi^*\psi_*)^2$ with the counit $\epsilon_\psi : \psi^*\psi_* \longrightarrow Id_{S^\sim - mod}$. Here η_ψ and ϵ_ψ are adjunction morphisms. This comonad structure induces a comultiplication

$$\delta'_\psi : S^\sim \otimes_{R^\sim} S^\sim \longrightarrow (S^\sim \otimes_{R^\sim} S^\sim) \otimes_{S^\sim} (S^\sim \otimes_{R^\sim} S^\sim) \simeq S^\sim \otimes_{R^\sim} S^\sim \otimes_{R^\sim} S^\sim$$

on the $S^\sim$-bimodule $S^\sim \otimes_{R^\sim} S^\sim$. The pair $(\delta_{S^\sim}, S^\sim \otimes_{R^\sim} S^\sim)$ is a coalgebra in the monoidal category $S^\sim - bimod^\sim = (S^\sim - bimod, \otimes_{S^\sim}, S^\sim)$ of $S^\sim$-bimodules with the counit ϵ'_ψ : $S^\sim \otimes_{R^\sim} S^\sim \longrightarrow S^\sim$ induced by the multiplication $S^\sim \otimes S^\sim \longrightarrow S^\sim$. The inverse image functor $\psi^* = S^\sim \otimes_{R^\sim} : R^\sim - mod \longrightarrow S^\sim - mod$ is the composition of the canonical functor

$$R^\sim - mod \longrightarrow (S^\sim \otimes_{R^\sim} S^\sim) - comod, \quad M \longmapsto (\psi^*(M), \delta_\psi(M)) = (S^\sim \otimes_{R^\sim} M, \delta'_\psi), \quad (1)$$

and the forgetful functor

$$(S^\sim \otimes_{R^\sim} S^\sim) - comod \longrightarrow S^\sim - mod, \quad (N, \nu : N \longrightarrow (S^\sim \otimes_{R^\sim} S^\sim) \otimes_{S^\sim} N) \longmapsto N \quad (2)$$

4.2.1.2. Proposition. *Suppose the category $\mathcal{A}$ has kernels of coreflexive pairs of arrows. Then the functor (1) is an equivalence of categories if*

$$\{\mathbf{Spec}_{\mathcal{A}^\sim}(\psi) : \mathbf{Spec}_{\mathcal{A}^\sim}(S^\sim) \longrightarrow \mathbf{Spec}_{\mathcal{A}^\sim}(R^\sim)\}$$

is an **fp** *cover.*

Proof. The assertion follows from (the dual form of) the Barr-Beck theorem. Details are left to the reader. ∎

4.2.2. The fpqc quasi-topology on affine schemes. We define **fpqc** *covers* as sets of *flat* morphisms $\mathbf{U} = \{u_i : U_i \longrightarrow X | i \in J\}$ such that for a finite subset I of J, the set of morphisms $\mathbf{U} = \{\mathbf{Spec}_{\mathfrak{C}}(u_i) \ | i \in I\}$ is conservative. (i.e. it reflects isomorphisms). Denote by $\tau^{\mathfrak{C}}_{\mathbf{fpqc}}$ the function which assigns to each object X of the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes in $\mathcal{A}^\sim$ the family of **fpqc** covers of X. One can see that $\tau^{\mathfrak{C}}_{\mathbf{fpqc}}$ is a quasi-topology.

4.2.3. The lqc quasi-topology. We define **lqc** covers as **fpqc** covers $\mathfrak{U} = \{u_i : U_i \longrightarrow X | i \in J\}$ such that all inverse image functors $\mathbf{Spec}_{\mathfrak{C}}(u_i)^*$ are localizations. We denote by $\tau^{\mathfrak{C}}_{\mathbf{lqc}}$ the function $X \longmapsto \{\mathbf{lqc}$ covers of $X\}$.

4.2.3.1. Proposition. *Suppose the category $\mathcal{A}$ has countable colimits which are preserved by functors $M \otimes -$ for all $M \in Ob\mathcal{A}$ and Φ is a functor $\mathcal{A} \longrightarrow End_\omega C$. Then the* **lqc** *quasi-topology is a pretopology.*

Proof. It suffices to prove the assertion in the case when $\mathcal{A}^\sim = End^\sim_\omega C$ and $\Phi^\sim$ acting naturally on C. Let $\{u_i : U_i \longrightarrow X | \ i \in J\}$ be a finite Zariski cover and $f : Y \longrightarrow X$ an arbitrary morphism of Aff_ω/C. By 2.6.3.1, the canonical morphism $p_i : U_i \times_C Y \longrightarrow Y$ is a localization for any $i \in J$. The conservativity of $\{u_i^* : X \longrightarrow U_i| \ i \in J\}$ implies that of $\{p_i^* : Y \longrightarrow U_i \times_C Y| \ i \in J\}$. ∎

4.2.3.2. The lpq quasi-topology on $\mathbf{Aff}_{\mathcal{A}^\sim}$. A finite set of arrows

$$\{\mathbf{Spec}_{\mathcal{A}^\sim}(S_i^\sim) \longrightarrow \mathbf{Spec}_{\mathcal{A}^\sim}(R^\sim)| \ i \in J\}$$

is an **lpq** cover iff it is conservative, all functors $S_i^\sim \otimes_{R^\sim}$ are exact, and the canonical morphism $S_i^\sim \otimes_{R^\sim} S_i^\sim \longrightarrow S_i^\sim$ is an isomorphism for all $i \in J$.

4.3. Descent. Let $(A, \mathfrak{T})$ be a quasi-site (i.e. $\mathfrak{T}$ is a quasi-topology on the category A), and let $F : A \longrightarrow B$ be a functor. We call the pair $(\mathfrak{T}, F)$ *subcanonical* if for any $X \in ObB$, the presheaf $h_X^F := B(F-, X)$ is a sheaf. Thus the pair $(\mathfrak{T}, Id_A)$ is subcanonical iff the quasi-topology $\mathfrak{T}$ is subcanonical in the sense of 4.1.5.2.

4.3.1. Lemma. *Suppose the functor $F : A \longrightarrow B$ is a composition of a functor $F' : A \longrightarrow B'$ and a fully faithful functor $G : B' \longrightarrow B$. If $(\mathfrak{T}, F)$ is subcanonical, then the pair $(\mathfrak{T}, F')$ is subcanonical. In particular, if $(\mathfrak{T}, F)$ is subcanonical and F is fully faithful, the quasi-topology $\mathfrak{T}$ is subcanonical.*

Proof is left to the reader. ∎

4.3.2. Proposition *Suppose the category $\mathcal{A}$ has countable colimits which are preserved by functors $M \otimes -$ for all $M \in Ob\mathcal{A}$ as well as by the functor $\Phi : \mathcal{A} \longrightarrow End_2 C$ (in particular it takes values in the subcategory $End_\omega C$). Let τ denote the* **fp** *quasi-topology $\tau^{\mathfrak{C}}_{\mathbf{fp}}$ on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes. And let F be the canonical functor $\mathbf{Aff}_{\mathcal{A}^\sim} \longrightarrow RCat/C$. Then the pair (τ, F) is subcanonical. In particular, the quasi-topology $\tau = \tau^{\mathfrak{C}}_{\mathbf{fp}}$ is subcanonical.*

Proof. It suffices to prove the assertion in the case when $\mathcal{A}^\sim = End^\sim_\omega C$ and $\Phi^\sim$ being a natural action of $\mathcal{A}^\sim$ on C.

(a) Let $\mathbb{G} = (G, \mu)$ be a monad on X such that the functor G preserves countable colimits, monomorphisms and kernels of pairs of morphisms and reflects isomorphisms. Then the square

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\eta} & \mathbb{G} \\ \eta \big\downarrow & & \big\downarrow \phi \\ \mathbb{G} & \xrightarrow{\psi} & \mathbb{G} \star \mathbb{G} \end{array} \qquad (1)$$

is cartesian. Here $\mathbf{1}$ is the identical functor Id_X regarded as a monad on X; ϕ and ψ denote the canonical monad morphisms.

In fact, since the functor G preserves kernels of pairs of morphisms and reflects isomorphisms, the square

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\eta} & G \\ \eta \big\downarrow & & \big\downarrow G\eta \\ G & \xrightarrow{\eta G} & G \circ G \end{array} \qquad (2)$$

is cartesian. Note that the canonical morphism $G \circ G \longrightarrow G \star G$ is (under the conditions on G) a monomorphism. Therefore the square (1) is cartesian.

(b) Let U denote the category $\mathbb{G}-mod_X$ of $\mathbb{G}$-modules. The square (1) being cartesian implies that the square

$$\begin{array}{ccc} U \times_X U & \xrightarrow{p} & U \\ q \big\downarrow & & \big\downarrow \phi \\ U & \xrightarrow{\phi} & X \end{array} \qquad (3)$$

is cocartesian. Here ϕ denote the morphism having the forgetful functor as an inverse image functor.

(c) Let $f, g : U \rightrightarrows Y$ be morphisms such that $f \circ p = g \circ q$. This means that $p^* f^* \simeq q^* g^*$. It follows from (b) that for any $L \in ObY$, there exists an object M of X defined uniquely up to isomorphism and such that $f^*(L) \simeq \phi^*(M) \simeq g^*(L)$. The map $L \longmapsto M$ extends to a functor γ^* such that $f^* \simeq \phi^* \gamma^* \simeq g^*$. The functor γ^* is defined uniquely up to isomorphism and is right exact. The latter follows from the fact that $\phi^* \gamma^*$ has this property and the functor ϕ^* preserves kernels of pairs of morphisms and monomorphisms and reflects isomorphisms. Therefore γ^* is an inverse image functor of a uniquely defined morphism $\gamma : Y \longrightarrow X$. ∎

4.3.3. Corollary. *Let the assumptions of 4.3.2 hold. Then for any* $\mathbf{X} = (X, f) \in ObRCat/C$, *the functor*

$$h^o_{\mathbf{X}} : \mathbf{Aff}^{op}_{\mathcal{A}^\sim} \longrightarrow \mathbf{Sets}, \quad \mathbf{Y} \longmapsto (RCat/C)(\mathbf{Spec}_{\mathfrak{C}}(\mathbf{Y}), \mathbf{X}),$$

is a sheaf on $(\mathbf{Aff}_{\mathcal{A}^\sim}, \mathfrak{T}_{\mathbf{fp}})$.

Proof. The assertion follows from 4.3.1. and 4.3.2. ∎

4.4. Coinduced topologies. Let $\mathfrak{T}$ be a quasi-topology on a category A. For any $V \in ObA$, denote by $\mathfrak{T}(V)$ the set of sieves which contain a $\mathfrak{T}$-cover of V. Let $F : A \longrightarrow B$ be a functor. For any $V \in ObA$, the functor F induces a functor $F_V : A/V \longrightarrow B/F(V)$. Denote by $\mathfrak{T}_F(F(V))$ the family of all sieves S in $B/F(V)$ such that $F_V^{-1}(S) \in \mathfrak{T}(V)$, i.e. $F_V^{-1}(S)$ contains a $\mathfrak{T}$-cover of V. For an arbitrary $X \in ObB$, denote by $\mathfrak{T}_F(X)$ the family of all sieves S of B/X such that for any morphism $v : F(V) \longrightarrow X$, $S^v \in \mathfrak{T}_F(F(V))$.

4.4.1. Proposition. *The map $X \longmapsto \mathfrak{T}_F(X)$, $X \in ObB$, defines a topology, $\mathfrak{T}_F$, on the category B.*

Proof. (a) Let $f : Y \longrightarrow X$ be a morphism of B, and let $R \in \mathfrak{T}_F(X)$. By definition of $\mathfrak{T}_F$, for any $u : F(U) \longrightarrow Y$, $(R^f)^u = R^{fu} \in \mathfrak{T}_F(F(U))$. Hence $R^f \in \mathfrak{T}_F(Y)$.

(b) Let $R \in \mathfrak{T}_F(X)$, and let R_1 be a sieve of B/X such that for any $(Y \xrightarrow{f} X) \in ObR$, the sieve R_1^f belongs to $\mathfrak{T}_F(Y)$. We claim that $R_1 \in \mathfrak{T}_F(X)$.

In fact, let $u : F(U) \longrightarrow X$ be an arbitrary morphism. We need to show that $R_1^u \in \mathfrak{T}_F(F(U))$, i.e. $F^{-1}(R_1^u) \in \mathfrak{T}(U)$.

Since $R \in \mathfrak{T}_F(X)$, $R^u \in \mathfrak{T}_F(F(U))$, i.e. $F^{-1}(R^u) \in \mathfrak{T}(U)$. By assumption for any $(V \xrightarrow{\xi} U) \in F^{-1}(R^u)$, $F^{-1}((R_1^u)^{F\xi}) \in \mathfrak{T}(V)$ which implies that $(F^{-1}(R_1^u))^{\xi} \in \mathfrak{T}(V)$, hence $F^{-1}(R^u) \in \mathfrak{T}(U)$. ■

The topology $\mathfrak{T}_F$ will be called *the topology coinduced by* $(\mathfrak{T}, F)$. The following assertion gives a description of the coinduced topology $\mathfrak{T}_F$ in the case when the topology $\mathfrak{T}$ is defined by a continuous base.

4.4.2. Lemma. *Let τ be a continuous base of the topology $\mathfrak{T}$ (cf. A.1.5.1).*

(a) A family of morphisms $\{u_i : U_i \longrightarrow X | i \in J\}$ of B is a $\mathfrak{T}_F$-cover of X iff for any morphism $v : F(V) \longrightarrow X$, there exists a subset J^v of J and a set of commutative diagrams

$$\begin{array}{ccc} F(U_{ij}) & \xrightarrow{u_{ij}} & U_i \\ {\scriptstyle F(p_{ij})}\downarrow & & \downarrow{\scriptstyle u_i} \\ F(V) & \xrightarrow{v} & X \end{array} \qquad j \in J_i \qquad (1)$$

such that the set of arrows $\{p_{ij} : U_{ij} \longrightarrow V | i \in J^v,\ j \in J_i\}$ belongs to τ.

(b) The topology $\mathfrak{T}_F$ is the finest among the topologies $\mathfrak{T}'$ on B having the following property:

(#) For any $V \in ObA$ and any cover $\{U_i \longrightarrow F(V) |\ i \in J\}$ in $\mathfrak{T}'$, there exists a subset I of J and a set of commutative diagrams

$$\begin{array}{ccc} F(U_{ij}) & \xrightarrow{u_{ij}} & U_i \\ {\scriptstyle F(p_{ij})}\downarrow & & \downarrow{\scriptstyle u_i} \\ F(V) & \xrightarrow{id_{F(V)}} & F(V) \end{array} \qquad j \in J_i \qquad (1)$$

such that the set of arrows $\{p_{ij} : U_{ij} \longrightarrow V | i \in J,\ j \in J_i\}$ belongs to τ.

Proof. The assertion follows from the definition of the topology $\mathfrak{T}_F$ and A.1.5.2. Details are left to the reader. ■

4.4.3. The site $(B_F, \mathfrak{T}_F)$. Fix a site $(A, \mathfrak{T})$ and a functor $F : A \longrightarrow B$. For an object X of B, denote by F_X the functor $F/X \longrightarrow B$, $(V, F(V) \xrightarrow{h} X) \longmapsto F(V)$ and by $F \downarrow X$ the natural cone $F_X \longrightarrow X$. Let B_F denote the full subcategory of the category B generated by all objects X of B such that the cone $F \downarrow X$ is initial. Note that the functor

$$B \longrightarrow Funct(A^{op}, \mathbf{Sets}), \quad X \longmapsto B(F-, X), \tag{1}$$

induces a fully faithful functor from the category B_F to the category $A^\wedge$ of functors from A^{op} to **Sets**. In fact, for any two objects X_1, X_2 of the category B, any functor morphism $\psi : B(F-, X_1) \longrightarrow B(F-, X_2)$ induces a cone morphism

$$F_\psi : F \downarrow X_1 \longrightarrow F \downarrow X_2, \quad (F(V) \xrightarrow{\xi} X_1) \longmapsto (F(V) \xrightarrow{\psi(\xi)} X_2)$$

If the cone $F \downarrow X_1$ is initial, F_ψ induces a uniquely defined morphism $X_1 \longrightarrow X_2$.

4.4.3.1. Lemma. *Suppose the pair $(\mathfrak{T}, F : A \longrightarrow B)$ is subcanonical. Then the topology induced on the subcategory B_F by the topology $\mathfrak{T}_F$ is subcanonical.*

Proof. (a) Note that a topology $\mathfrak{T}'$ on a category A' is subcanonical iff for any $X \in ObA'$ and any $R \in \mathfrak{T}'(X)$, the cone R is initial.

In fact, $\mathfrak{T}'$ is subcanonical iff for any $Y \in ObA'$ and any $R \in \mathfrak{T}'(X)$, the canonical morphism $A'(X, Y) \longrightarrow limA'(\mathcal{F}_X|R, Y)$ is an isomorphism. Here $\mathcal{F}_X|R$ is the restriction to R of the forgetful functor $\mathcal{F}_X : A/X \longrightarrow A$. This means that any cone $R \longrightarrow Y$ decomposes into the canonical cone $R = (F_X|R \longrightarrow X)$ and a unique morphism $X \longrightarrow Y$, hence the cone R is initial.

(b) Let $R \in \mathfrak{T}_F(X)$, i.e. for any $u : F(V) \longrightarrow X$, $R_F^u := F^{-1}(R^u) \in \mathfrak{T}(V)$. By assumption, $B(F-, X)$ is a sheaf on $(A, \mathfrak{T})$. In particular, the canonical morphism $B(F(V), X) \longrightarrow limB(F \circ \mathcal{F}_V|R_F^u, X)$ is an isomorphism. This means that the cone $F(R_F^u) \subset B/F(V)$ is initial. If the cone $F \downarrow X$ is initial, this implies that the cone R is initial. The assertion follows now from (a). ■

4.4.4. The topology coinduced by a quasi-topology. Fix a quasi-site (A, τ). Applying the construction of a coinduced topology to the identical functor $A \longrightarrow A$, we obtain a *topology, $\mathfrak{T}^\tau$, coinduced by the quasi-topology* $\mathfrak{T}$. It follows from 4.1.3.4 that if τ is continuous (e.g. it is a pretopology), then $\mathfrak{T}^\tau$ coincides with the coarsest topology $\mathfrak{T}_\tau$ generated by τ (cf. 4.1.3.3). If τ is subcanonical, then by 4.4.3.1, the topology $\mathfrak{T}^\tau$ is subcanonical.

4.4.5. Canonical topologies on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$. These are topologies coinduced by the canonical quasi-topologies introduced in 4.2.1, 4.2.2, and 4.2.3.

We denote by $\mathfrak{T}^{\mathfrak{C}}_{\mathbf{fp}}$ the topology coinduced by $\tau^{\mathfrak{C}}_{\mathbf{fp}}$.

We denote by $\mathfrak{T}^{\mathfrak{C}}_{\mathbf{fpqc}}$ the topology on $\mathbf{Aff}_{\mathcal{A}^\sim}$ coinduced by **fpqc** covers and call it **fpqc** *topology.* Clearly $\tau^{\mathfrak{C}}_{\mathbf{fpqc}}$ is finer than the quasi-topology $\tau^{\mathfrak{C}}_{\mathbf{fp}}$ (see 4.2.1), hence the topology $\mathfrak{T}^{\mathfrak{C}}_{\mathbf{fpqc}}$ is finer than the topology $\mathfrak{T}^{\mathfrak{C}}_{\mathbf{fp}}$.

We denote the topology coinduced by **lqc** covers by $\mathfrak{T}_{\mathbf{lqc}}$. Under conditions of 4.2.3.1, **lqc** covers form a pretopology. Hence in this case, $\mathfrak{T}_{\mathbf{lqc}}$ coincides with the associated topology.

4.5. The category of $(\mathfrak{T}|F)$-covers. Fix a quasi-site $(A,\mathfrak{T})$ and a functor $F : A \longrightarrow B$ such that the pair $(F,\mathfrak{T})$ is subcanonical. Let $X \in ObB$. For any set $\mathbf{U} = \{(U_i, F(U_i) \xrightarrow{u_i} X) \mid i \in J\}$ of objects of the category F/X, denote by $R_{\mathbf{U}}$ the full subcategory of the category F/X objects of which are all objects of F/X having a morphism to some of the objects of $\mathbf{U}$.

We define the category of $Cov_{F|\mathfrak{T}}$ of $F|\mathfrak{T}$-*covers* as follows: objects of $Cov_{F|\mathfrak{T}}$ are pairs $(\mathbf{U},X)$, where $\mathbf{U}$ is a set of objects of F/X such that $X = colim(F|R_{\mathbf{U}})$. Here $F|R_{\mathbf{U}}$ denotes the restriction to $R_{\mathbf{U}}$ of the canonical functor $F/X \longrightarrow B$, $(U, F(U) \longrightarrow X) \longmapsto F(U)$. A morphism from $(\mathbf{U},X)$ to $(\mathbf{U}',X')$ is given by a morphism $f : X \longrightarrow X'$ such that the sieve $R^f_{\mathbf{U}'}$ of F/X contains the sieve $R_{\mathbf{U}}$. Here $R^f_{\mathbf{U}'}$ is the full subcategory of F/X generated by all objects (V,h) such that $(V, f\circ h) \in ObR_{\mathbf{U}'}$.

4.5.1. Locally affine $(\mathfrak{T}|F)$-covers. Fix an object X of the category B and a pair of objects $(U, F(U) \xrightarrow{u} X)$, $(V, F(V) \xrightarrow{v} X)$ of the category F/X. Let $D^F_{(U,u;V,v)}$ denote the category objects of which are triples (ϕ, W, ψ), where $W \in ObA$ and $W \xrightarrow{\phi} U$, $W \xrightarrow{\psi} V$ are morphisms of the subcategory A^{τ} (see 4.1.2.1) such that $u \circ F\phi = v \circ F\psi$. Morphisms of triples are defined in an obvious way.

We call a cover $(\mathbf{U},X)$, $\mathbf{U} = \{(U_i, F(U_i) \xrightarrow{u_i} X) \mid i \in J\}$, *locally affine* if for any $i, j \in J$, there exists a set of objects $\mathbf{U}_{ij}$ of the category $D^F_{(U_i,u_i;U_j,u_j)}$ such that

(#) the diagram $D^{u_i,u_j} \longrightarrow B$ has a colimit and the cone $F|R_{\mathbf{U}_{ij}} \longrightarrow colim(D^{u_i,u_j})$ is initial (i.e. $colim(F|R_{\mathbf{U}_{ij}})$ exists and is naturally isomorphic to $colim(D^{u_i,u_j})$).

Informally speaking, a locally affine $(\mathfrak{T}|F)$-cover is generated by "elements" of the quasi-site $(A,\mathfrak{T})$ with relations from $(A,\mathfrak{T})$.

We denote by $\mathfrak{Lac}_{\mathfrak{T}|F}$ the full subcategory of the category $Cov_{\mathfrak{T}|F}$ objects of which are locally affine $(\mathfrak{T}|F)$-covers.

4.5.2. Zariski $(\mathfrak{T}|F)$-covers. We call a cover $(\mathbf{U},X)$, $\mathbf{U} = \{(U_i, F(U_i) \xrightarrow{u_i} X) \mid i \in J\}$, a *Zariski* $(\mathfrak{T}|F)$-*cover* if all arrows u_i are monomorphisms and for any $i, j \in J$, there exists a set of objects $\mathbf{U}_{ij} = \{(u^{i\nu}_{ij}, U^{\nu}_{ij}, u^{i\nu}_{ij}) | \, \nu \in J_{ij}\}$ of the category $D^F_{(U_i,u_i;U_j,u_j)}$ satisfying the condition (#) of 4.5.1 and having the property: for any $\nu \in J_{ij}$, any pair of morphisms $g_1, g_2 : W \longrightarrow U^{\nu}_{ij}$ such that $u^{i\nu}_{ij} \circ g_1 = u^{i\nu}_{ij} \circ g_2$ and $u^{j\nu}_{ij} \circ g_1 = u^{j\nu}_{ij} \circ g_2$ is trivial, i.e. $g_1 = g_2$.

We denote by $\mathfrak{Zar}_{\mathfrak{T}|F}$ the full subcategory of the category $Cov_{\mathfrak{T}|F}$ objects of which are Zariski $(\mathfrak{T}|F)$-covers. It follows that $\mathfrak{Zar}_{\mathfrak{T}|F}$ is a subcategory of the category $\mathfrak{Lac}_{\mathfrak{T}|F}$ of locally affine $(\mathfrak{T}|F)$-covers.

4.5.3. Canonical functors. We shall write $\mathfrak{T}$ instead of $(\mathfrak{T}|Id_A)$. Thus we have the category $Cov_{\mathfrak{T}}$ of $\mathfrak{T}$-covers and its subcategories $\mathfrak{Lac}_{\mathfrak{T}}$ and $\mathfrak{Zar}_{\mathfrak{T}}$ of resp. locally affine and Zariski $\mathfrak{T}$-covers. Since the pair $(F,\mathfrak{T})$ is subcanonical, the functor F induces a functor $F_{Cov_{\mathfrak{T}}} : Cov_{\mathfrak{T}} \longrightarrow Cov_{(\mathfrak{T}|F)}$.

We have a canonical fully faithful functor $A \longrightarrow \mathfrak{Zar}_{\mathfrak{T}}$, $U \longmapsto (U \xrightarrow{id_U} U)$.

The composition of these two functors gives a canonical functor $F_{\mathfrak{T}}$ from the category A to the category $\mathfrak{Zar}_{\mathfrak{T}|F}$ which maps any object U of A to the cover $\{F(U) \overset{id_{F(U)}}{\longrightarrow} F(U)\}$.

4.6. $(\mathfrak{T}|F)$**-spaces.** Let $(\mathbf{U}, X)$, $\mathbf{U} = \{(U_i, F(U_i) \overset{u_i}{\longrightarrow} X) \mid i \in J\}$, be a cover. Suppose that for any $i \in J$, we are given a $\mathfrak{T}$-cover $\{U_{ij} \overset{u_{ij}}{\longrightarrow} U_i | \ j \in J_i\}$. We call a set of objects $\{(U_{ij}, F(U_{ij}) \overset{u_i \circ F(u_{ij})}{\longrightarrow} X) \mid i \in J,\ j \in J_i\}$ a *refinement*, or a *$\mathfrak{T}$-refinement of* $\mathbf{U}$. Since the pair $(F, \mathfrak{T})$ is subcanonical, any refinement of a cover is a cover.

We say that two covers $(\mathbf{U}, X)$ and $(\mathbf{U}', X)$ are *$\mathfrak{T}$-equivalent* if they have a common refinement. It follows that $\mathfrak{T}$-equivalence is an equivalence relation. We define the category $Es_{(\mathfrak{T}|F)}$ of *$(\mathfrak{T}|F)$-spaces* as follows. Objects of $Es_{(\mathfrak{T}|F)}$ are pairs $(X, \mathfrak{t})$, where $\mathfrak{t}$ is an equivalence class of covers of X. Morphisms from $(X, \mathfrak{t})$ to $(X', \mathfrak{t}')$ are given by morphisms $f : X \longrightarrow X'$ such that for any $(\mathbf{U}', X') \in \mathfrak{t}'$, there exists $(\mathbf{U}, X) \in \mathfrak{t}$ such that f defines a morphism $(\mathbf{U}, X) \longrightarrow (\mathbf{U}', X')$ of covers. One can see that if $(X, \mathfrak{t}) \overset{f}{\longrightarrow} (X', \mathfrak{t}') \overset{g}{\longrightarrow} (X'', \mathfrak{t}'')$ are morphisms, then $(X, \mathfrak{t}) \overset{gf}{\longrightarrow} (X'', \mathfrak{t}'')$ is a morphism too.

4.6.1. Proposition. *Suppose the quasi-topology $\mathfrak{T}$ is continuous (for instance, $\mathfrak{T}$ is a pretopology). Then we have a natural functor*

$$Cov_{(\mathfrak{T}|F)} \longrightarrow Es_{(\mathfrak{T}|F)} \tag{1}$$

which is equivalent to the localization of the category $Cov_{(\mathfrak{T}|F)}$ by the class of morphisms of the form $(\mathbf{U}, X) \overset{id_X}{\longrightarrow} (\mathbf{U}', X)$.

Proof is left to the reader. ■

4.6.2. Locally affine $(\mathfrak{T}|F)$**-spaces.** We say that two locally affine $(\mathfrak{T}|F)$-covers $(\mathbf{U}, X)$ and $(\mathbf{U}', X)$ are *$\mathfrak{T}$-equivalent* if they have a common locally affine refinement. We define the category $\mathfrak{Las}_{(\mathfrak{T}|F)}$ of *locally affine $(\mathfrak{T}|F)$-spaces* as follows. Objects of $\mathfrak{Las}_{(\mathfrak{T}|F)}$ are pairs $(X, \mathfrak{t})$, where $\mathfrak{t}$ is an equivalence class of locally affine $(\mathfrak{T}|F)$-covers of X. Morphisms from $(X, \mathfrak{t})$ to $(X', \mathfrak{t}')$ are given by morphisms $f : X \longrightarrow X'$ such that for any locally affine cover $(\mathbf{U}', X') \in \mathfrak{t}'$, there exists a locally affine cover $(\mathbf{U}, X) \in \mathfrak{t}$ such that f defines a morphism $(\mathbf{U}, X) \longrightarrow (\mathbf{U}', X')$ of covers. The composition is defined as in $Es_{(\mathfrak{T}|F)}$. We have an analog of Proposition 4.6.1:

4.6.2.1. Proposition. *Suppose the quasi-topology $\mathfrak{T}$ is continuous. Then we have a natural functor*

$$\mathfrak{Lac}_{(\mathfrak{T}|F)} \longrightarrow \mathfrak{Las}_{(\mathfrak{T}|F)} \tag{1}$$

which is equivalent to the localization of the category $\mathfrak{Lac}_{(\mathfrak{T}|F)}$ of locally affine covers by the class of morphisms of the form $(\mathbf{U}, X) \overset{id_X}{\longrightarrow} (\mathbf{U}', X)$. In particular, there is a canonical functor

$$F_{\mathfrak{T}} : \mathcal{A} \longrightarrow Es_{(\mathfrak{T}|F)}. \tag{2}$$

4.6.2.2. Note. In the case of an arbitrary (i.e. not necessarily continuous) quasi-topology $\mathfrak{T}$, there is a canonical functor

$$F_{\mathfrak{T}} : \mathcal{A}_{\mathfrak{T}} \longrightarrow Es_{(\mathfrak{T}|F)} \tag{3}$$

(cf. 4.1.2.2). In particular, there is a canonical functor

$$\mathcal{A}^{\mathfrak{T}} \longrightarrow Es_{(\mathfrak{T}|F)} \tag{4}$$

(cf. 4.1.2.1).

4.6.3. $(\mathfrak{T}|F)$**-schemes.** We define a $(\mathfrak{T}|F)$-*scheme* as a pair $(X, \mathfrak{t})$, where $\mathfrak{t}$ is an equivalence class (with respect to refinements) of Zariski $(\mathfrak{T}|F)$-covers. Morphisms from $(X, \mathfrak{t})$ to $(X', \mathfrak{t}')$ are given by morphisms $f : X \longrightarrow X'$ such that for any Zariski $(\mathfrak{T}|F)$-cover $(\mathbf{U}', X') \in \mathfrak{t}'$, there exists a Zariski $(\mathfrak{T}|F)$-cover $(\mathbf{U}, X) \in \mathfrak{t}$ such that f defines a morphism $(\mathbf{U}, X) \longrightarrow (\mathbf{U}', X')$ of covers. We denote the category of $(\mathfrak{T}|F)$-schemes by $\mathfrak{Sch}_{\mathfrak{T}|F}$.

We leave to the reader the formulation of the corresponding version of Propositions 4.6.1 and 4.6.2.1.

4.6.4. Quasi-separated locally affine $(\mathfrak{T}|F)$**-spaces and schemes.** We call a locally affine $(\mathfrak{T}|F)$-space (resp. $(\mathfrak{T}|F)$-scheme) $(X, \mathfrak{t})$ *quasi-separated* if it has a (resp. Zariski) $\mathfrak{T}$-cover $\{F(U_i) \longrightarrow X |\ i \in J\}$ of $\mathbf{X}$ such that for any $i, j \in J$, the set of objects $\mathbf{U}_{ij}$ in the condition (#) in 4.5.1 (resp. in the conditions of 4.5.2) is finite.

4.6.5. Semiseparated locally affine $(\mathfrak{T}|F)$**-spaces and schemes.** We call a $\mathfrak{T}$-cover $\{F(U_i) \longrightarrow X |\ i \in J\}$ of a $(\mathfrak{T}|F)$-space $(X, \mathfrak{t})$ *semiseparated* if for any $i, j \in J$, there exists a cartesian product of the form

$$\begin{array}{ccc} F(U_{ij}) & \xrightarrow{F(p_{ij})} & F(U_i) \\ {\scriptstyle F(q_{ij})}\downarrow & & \downarrow \\ F(U_j) & \longrightarrow & X \end{array}$$

A space $(X, \mathfrak{t})$ which has a semiseparated cover is called *semiseparated.* Clearly every semiseparated space is locally affine.

4.7. $(\mathfrak{T}|F)$**-spaces and** $\mathfrak{T}$**-spaces.** Suppose the quasi-topology $\mathfrak{T}$ on the category A is subcanonical. This means exactly that the pair $(\mathfrak{T}, h^A)$, where h^A is the canonical functor $A \longrightarrow Func(A^{op}, \mathbf{Sets})$, $U \longmapsto A(-, U)$, is subcanonical. We will write $\mathfrak{T}$-spaces instead of $(\mathfrak{T}|h^A)$-spaces.

4.7.1. Proposition. *Suppose the pair* $(\mathfrak{T}, F)$ *is subcanonical and* $\mathfrak{T}$ *is continuous (e.g. it is a pretopology). Then*

(a) For any $(\mathfrak{T}|F)$*-space* $(X, \mathfrak{t})$*, the presheaf*

$$Es_{\mathfrak{T}|F}(F_{\mathfrak{T}}-, (X, \mathfrak{t})) : A^{op} \longrightarrow \mathbf{Sets}$$

is a sheaf.

(b) The map

$$(X,\tau) \longmapsto Es_{\mathfrak{T}|F}(F_{\mathfrak{T}}(-),(X,\tau))$$

determines a functor $h_{\mathfrak{T}|F} : Es_{\mathfrak{T}|F} \longrightarrow Sheaves(A,\mathfrak{T})$. *The functor* $h_{\mathfrak{T}|F}$ *is faithful (resp. fully faithful) if* F *is faithful (resp. fully faithful).*

(c) The functor $h_{\mathfrak{T}|F}$ *maps locally affine* $(\mathfrak{T}|F)$*-spaces to locally affine* $\mathfrak{T}$*-spaces and* $(\mathfrak{T}|F)$*-schemes to* $\mathfrak{T}$*-schemes.*

Proof. (a) This follows from fact that the pair $(\mathfrak{T},F)$ is subcanonical.

(b) The assertion follows from (a) and definitions.

(c) Fix a $(\mathfrak{T}|F)$-space $(X,\mathfrak{t})$. To the space $(X,\mathfrak{t})$ there corresponds the $(\mathfrak{T}|h^{\mathfrak{T}})$-space $(\mathcal{F},\mathfrak{t}^{\sim})$, where $h^{\mathfrak{T}}$ is the canonical functor $A \longrightarrow Sheaves(A,\mathfrak{T})$, $\mathcal{F}$ denotes the sheaf $Es_{\mathfrak{T}|F}(F_{\mathfrak{T}}-,(X,\mathfrak{t}))$ (cf. (a)), and $\mathfrak{t}^{\sim}$ is the class of equivalent sieves of $h^{\mathfrak{T}}/\mathcal{F}$ corresponding to $\mathfrak{t}$. The $(\mathfrak{T}|h^{\mathfrak{T}})$-space $(\mathcal{F},\mathfrak{t}^{\sim})$ has the following property: for any $(Z,f) \in Obh^{\mathfrak{T}}/\mathcal{F}$ and any $\mathbf{U} \in \mathfrak{t}^{\sim}$, the sieve $R_{\mathbf{U}}^{(Z,f)}$ contains a sieve $R_{\mathbf{U}'}$ for some $\mathfrak{T}$-cover $\mathbf{U}'$ of Z. The rest of the argument is left to the reader. ■

4.7.2. Note that the functor $h_{\mathfrak{T}|F}$ of 4.7.1 maps quasi-separated (resp. semiseparated) locally affine $(\mathfrak{T}|F)$-spaces to quasi-separated (resp. semiseparated) $\mathfrak{T}$-spaces. This correspondence induces the similar correspondence between quasi-separated (resp. semiseparated) $(\mathfrak{T}|F)$- and $\mathfrak{T}$-schemes.

4.7.3. Representable sheaves. We shall say that a sheaf of sets $\mathfrak{F}$ on the site $(A,\mathfrak{T})$ is *representable by a* $(\mathfrak{T}|F)$*-space* $(X,\mathfrak{t})$ if $\mathfrak{F} \simeq Es_{\mathfrak{T}|F}(F_{\mathfrak{T}}-,(X,\mathfrak{t}))$. Since $(X,\mathfrak{t})$ is a colimit of the natural functor $F_{\mathfrak{T}}/(X,\mathfrak{t}) \longrightarrow Es_{\mathfrak{T}|F}$, the representing $(\mathfrak{T}|F)$-space (if any) is defined uniquely up to isomorphism.

4.8. Closed immersions.

4.8.1. Representable morphisms. Let $\mathcal{P}$ be a class of morphisms of the category C satisfying the following conditions:

(a) A composition of a morphism from $\mathcal{P}$ with any isomorphism belongs to $\mathcal{P}$.

(b) If $f : X \longrightarrow Y$ is a morphism from $\mathcal{P}$, then for any $g : Z \longrightarrow Y$, there exists a fibered product $X \times_Y Z$ and the projection $X \times_Y Z \longrightarrow Z$ belongs to $\mathcal{P}$.

Let F, G be presheaves of sets. A morphism $F \longrightarrow G$ is called *representable by a morphism of* $\mathcal{P}$ if for any $h_X \longrightarrow G$, the projection $F \times_G h_X \longrightarrow h_X$ is of the form h_u for a morphism $u \in \mathcal{P}$.

Denote by $\mathcal{P}^{\wedge}$ the class of all morphisms of $C^{\wedge}$ representable by morphisms of $\mathcal{P}$. Note that $\mathcal{P}^{\wedge}$ is invariant under the base change: if $F \longrightarrow G$ belongs to $\mathcal{P}^{\wedge}$ and $H \longrightarrow G$ is an arbitrary morphism, then the projection $H \times_G F \longrightarrow H$ belongs to $\mathcal{P}^{\wedge}$. Clearly a morphism $h_X \longrightarrow h_Y$ belongs to $\mathcal{P}^{\wedge}$ iff it is of the form h_w with $w \in \mathcal{P}$.

4.8.1.1. Lemma. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be classes of morphisms of the category* C *satisfying the conditions (a), (b). Then*

(i) The intersection $\mathcal{P} \cap \mathcal{Q}$ *has the properties (a) and (b).*

(ii) If $\mathcal{P}$ *is closed under the composition, then* $\mathcal{P}^{\wedge}$ *has the same property.*

Proof is left to the reader. ∎

4.8.1.2. Standard examples. 1) The class $\mathfrak{M} = \mathfrak{M}(C)$ of all monomorphisms has property 4.8.1(b) and is closed under the composition.

2) Same holds for the class $\mathcal{E}^u = \mathcal{E}^u(C)$ of universal epimorphisms. Recall that a morphism $f : X \longrightarrow Y$ is called a *universal epimorphism* if for any morphism $V \longrightarrow Y$, there exists a fibered product $X \times_Y V$ and the canonical projection $X \times_Y V \longrightarrow V$ is an epimorphism.

4.8.2. Strict monomorphisms and closed immersions. For the definition of a strict monomorphism, see 1.6. Recall that if the category C has fibered coproducts, then strict monomorphisms can be defined as morphisms $X \longrightarrow Y$ such that the diagram $X \longrightarrow Y \rightrightarrows Y \coprod_X Y$ is exact.

The composition of a strict monomorphism with an isomorphism is a strict monomorphism. If the category C has fiber products, then the class $\mathfrak{M}_s = \mathfrak{M}_s(C)$ of strict monomorphisms of the category C satisfies the condition 4.8.1(b) too.

In fact, consider the diagram

$$\begin{array}{ccccc} X \times_Y V & \xrightarrow{p_2} & V & & \\ p_1 \downarrow & & \downarrow g & & \\ X & \xrightarrow{f} & Y & \rightrightarrows & Z \end{array} \qquad (1)$$

where $Y \rightrightarrows Z$ is an arbitrary pair of arrows from the class Λ_f of arrows equalizing f. But then p_2 is a universal arrow equalizing all pairs $\Lambda_f \circ g := \{(u_1 g, u_2 g) | \ (u_1, u_2) \in \Lambda_f\}$.

4.8.2.1. Lemma. *(a) If the composition, gf, of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a strict monomorphism, then f is a strict monomorphism.*

(b) Any retraction is a strict monomorphism.

Proof. (a) If gf is a kernel of a pair of arrows $u_1, u_2 : Z \rightrightarrows V$, then f is a kernel of the pair $u_1 g, u_2 g : Y \rightrightarrows V$. The general case is left to the reader.

(b) Let $p : X \longrightarrow Y$ is a retraction, i.e. there exists a morphism $e : Y \longrightarrow X$ such that $ep = id_X$. Then p is a kernel of the pair $id_X, pe : X \rightrightarrows X$.

In fact, if $f : Y \longrightarrow X$ is a morphism equalizing the pair (id_X, pe), then $f = p \circ (ef)$; and this decomposition is unique because p is a monomorphism. ∎

4.8.2.2. Closed immersions of presheaves of sets. And let F, G be presheaves of sets on C. We call a morphism $F \longrightarrow G$ a *closed immersion* if it belongs to $\mathfrak{M}_s^\wedge$, i.e. if it is representable by a strict monomorphism. In particular a closed immersion $h_X \longrightarrow h_Y$ of representable functors is of the form h_u, where u is a strict monomorphism.

4.8.3. Example. Let C be the category $CAff/k$ of commutative affine schemes over $Spec(k)$. Then strict monomorphisms are exactly closed immersions of affine schemes. Let X and Y be arbitrary schemes identified with the corresponding sheaves of sets on the category $C = CAff/k$. Then a morphism $X \longrightarrow Y$ is a closed immersion in the sense of the definition 4.8.2.2 iff it is a closed immersion of schemes in the conventional sense.

This example shows in particular that a strict monomorphism of (pre)sheaves is not necessarily a closed immersion. For instance, if $X \longrightarrow Y$ is a scheme morphism, the diagonal morphism $f : X \longrightarrow X \times_Y X$ is a kernel of the natural pair of arrows $X \times_Y X \rightrightarrows X$, hence is a strict monomorphism of sheaves of sets. But it is a closed immersion (in the sense of 4.8.2.2) only if the scheme morphism f is separated.

4.8.4. Example. Let k be an arbitrary associative ring and C the category $\mathbf{Aff}/k$ of affine schemes over k. A morphism $Spec(f) : Spec(A) \longrightarrow Spec(B)$ is a strict monomorphism iff the diagram $B \prod_A B \rightrightarrows B \xrightarrow{f} A$ is exact. The latter means that A is the quotient of B by the two-sided ideal $Ker(f)$.

5. Schemes and locally affine spaces.

5.0. Fix a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$. We assume that $\mathcal{A}$ is a category with countable colimits and that the tensor product preserves countable colimits, as well as the functor $\Phi : \mathcal{A} \longrightarrow EndC$. These conditions imply that the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes over $\mathfrak{C}$ has fiber products (cf. 3.2.2).

In what follows we shall apply the formalism of Section 4 to the case when $(A, \mathfrak{T})$ is the site $(\mathbf{Aff}_{\mathcal{A}^\sim}, \mathfrak{T}^{\mathfrak{C}}_{\mathbf{fpqc}})$ of affine schemes over the base $\mathfrak{C}$ with the **fpqc** topology, B is the category $RCat/C$ of 'spaces' over C, and $F : A \longrightarrow B$ is the functor

$$\mathbf{Spec}_{\mathfrak{C}} : \mathbf{Aff}_{\mathcal{A}^\sim} \longrightarrow RCat/C, \quad \mathbf{X} \longmapsto (Qcoh_{\mathbf{X}/\mathfrak{C}}, f_C), \tag{1}$$

(cf. 3.5). In this case we shall write $Es_{\mathbf{fpqc}}/\mathfrak{C}$ instead of $Es_{\mathfrak{T}|F}$.

5.1. The category of fpqc locally affine spaces over $\mathfrak{C}$. This is by definition the category $\mathfrak{Las}_{\mathfrak{T}|F}$ of locally affine $(\mathfrak{T}|F)$-spaces, where $\mathfrak{T}$ is the **fpqc** topology, $\mathfrak{T}^{\mathfrak{C}}_{\mathbf{fpqc}}$, on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes (see 4.4.5), and F is the embedding 5.0(1). We denote this category by $\mathfrak{Las}_{\mathbf{fpqc}}/\mathfrak{C}$. There is a canonical functor $\mathfrak{Las}_{\mathbf{fpqc}}/\mathfrak{C} \longrightarrow RCat/C$, $(\mathbf{X}, \tau) \longmapsto \mathbf{X}$, and a canonical functor $\mathbf{Aff}_{\mathcal{A}^\sim} \longrightarrow \mathfrak{Las}_{\mathbf{fpqc}}/\mathfrak{C}$ which assigns to any $\mathfrak{C}$-affine scheme $\mathbf{X}$ the object $(\mathbf{X}, \mathfrak{T}_{\mathbf{fpqc}}(\mathbf{X}))$ of the category $Es_{\mathbf{fpqc}}/\mathfrak{C}$ (cf. 4.6).

5.2. The category of schemes over $\mathfrak{C}$. The category $Sch/\mathfrak{C}$ of schemes over $\mathfrak{C}$ is defined as the category $Sch_{\mathfrak{T}|F}$ of $(\mathfrak{T}|F)$-schemes, where $\mathfrak{T}$ is the **lqc** topology $\mathfrak{T}^{\mathfrak{C}}_{\mathbf{lqc}}$ (cf. 4.4.5).

5.2.1. Locally affine spaces and schemes over k. Let k be an associative ring and C the category $k-mod$ of left k-modules. In this case, locally affine spaces (resp. schemes) over C will be called locally affine k-spaces (resp. k-schemes). Let $(\mathbf{X}, \tau)$, $\mathbf{X} = (X, f)$, be an arbitrary locally affine k-space (resp. k-scheme); and let $\mathfrak{U} = \{u_i : U_i \longrightarrow X \mid i \in J\}$ be an **fpqc** affine cover (resp. Zariski affine cover) of $(\mathbf{X}, \tau)$. This means that the category U_i is naturally equivalent to the category of R_i-modules, where $R_i = \mathcal{B}_i(\mathcal{O}_i, \mathcal{O}_i)^o$, $\mathcal{O}_i$ is the 'structure sheaf' on U_i: $\mathcal{O}_i = u_i^*(k)$ (see 2.5).

Thus any **fpqc** locally affine space or scheme over $k - mod$ is *locally* (in the corresponding sense) the category of left modules over algebras over k.

5.3. Quasi-coherent presheaves.

5.3.1. Quasi-coherent presheaves on fpqc $\mathfrak{C}$-spaces. The notion of a quasi-coherent presheaf on an **fpqc** locally affine space over $\mathfrak{C}$ is a special case of the notion of a quasi-coherent presheaf in a fibered category (see 1.5 and 1.5). It is defined as follows. Let $\mathfrak{A}$ be the category objects of which are affine schemes over $\mathfrak{C}$ and morphisms are flat morphisms. Let $\mathcal{F}_{\mathbf{fpqc}}$ be the fibered category with the base $\mathfrak{A}$ and such that the fiber over any affine scheme over $\mathfrak{C}$ Y is the category of left $\mathcal{O}_Y$-modules. Consider the canonical functor $\mathfrak{J} : \mathfrak{A} \longrightarrow \mathfrak{Las}_{\mathfrak{C}}$. To any **fpqc** locally affine space $(\mathbf{X}, \tau)$ over $\mathfrak{C}$, there corresponds the fibered subcategory $\mathcal{F}_{\mathbf{fpqc}}^{(\mathbf{X},\tau)}$. The category $Qcoh_{(\mathbf{X},\tau)}$ of quasi-coherent presheaves on $(\mathbf{X}, \tau)$ is by definition the category of quasi-coherent presheaves of the fibered category $\mathcal{F}_{\mathbf{fpqc}}^{(\mathbf{X},\tau)}$ (cf. 1.5).

In a less formal way, this can be described as follows. Fix an **fpqc** locally affine $\mathfrak{C}$-space $(\mathbf{X}, \tau)$. Denote by C_τ the category objects of which are elements of τ-covers which and morphisms are given by elements of τ-covers too. A quasi-coherent presheaf $M^\sim$ assigns to any object $U \longrightarrow \mathbf{X}$ of C_τ an $\mathcal{O}_U$-module $M^\sim(U)$ and to any morphism $(U \longrightarrow \mathbf{X}) \xrightarrow{u} (V \longrightarrow \mathbf{X})$ of C_τ an isomorphism $\xi_u : u^*(M^\sim(V)) \longrightarrow M^\sim(U)$.

5.3.1.1. Note that for any **fpqc** locally affine space $(\mathbf{X}, \tau)$, $\mathbf{X} = (X, f)$, over $\mathfrak{C}$, the canonical functor $X \longrightarrow Qcoh_{(\mathbf{X},\tau)}$, $M \longmapsto M^\sim$, where $M^\sim(U \xrightarrow{u} X) = u^*(M)$, is faithful. This follows from the fact that for any cover $\mathfrak{U}$, the set of inverse image functors $\{u^* | \ (U \xrightarrow{u} X) \in \mathfrak{U}\}$ is conservative.

5.3.2. Quasi-coherent presheaves on a scheme over $\mathfrak{C}$. The category $Qcoh_{(\mathbf{X},\tau)}$ of quasi-coherent presheaves on a scheme $(\mathbf{X}, \tau)$ is described in a similar way, with the **fpqc** quasi-topology replaced by the Zariski (i.e. **lqc**) quasi-topology. As in the **fpqc** case, the canonical functor $X \longrightarrow Qcoh_{(\mathbf{X},\tau)}$ is faithful. Details are left to the reader.

5.4. Coproduct of schemes over $\mathfrak{C}$. Fix a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$.

5.4.1. Proposition. *Let $\mathcal{F} = \{f_i : X_i \longrightarrow C | i \in J\}$ be a finite family of schemes over $\mathfrak{C}$. Suppose the categories C and $\mathcal{A}$ have finite products. Then a coproduct of the family $\mathcal{F}$ in $RCat/C$ exists and is a scheme over $\mathfrak{C}$.*

Proof. By 1.3.1, there exists a coproduct of $\mathcal{F}$ in $RCat_C$. If $(u_{ij} : U_{ij} \longrightarrow X_i | j \in J_i)$ is an affine cover of (X_i, f_i), then $(u_{ij} : U_{ij} \longrightarrow X_i | i \in J, \ j \in J_i)$ is an affine cover of $\coprod_{i\in J}(X_i, f_i)$. ∎

5.4.2. Corollary. *If the categories C and $\mathcal{A}$ have finite products, then the category $Sch_{\mathfrak{C}}$ of schemes over $\mathfrak{C}$ has finite coproducts.*

5.5. Noncommutative locally affine spaces and schemes as sheaves of sets. Let $(A, \mathfrak{T})$ be the site $(\mathbf{Aff}_{\mathcal{A}^\sim}, \mathfrak{T}^{\mathfrak{C}}_{\mathbf{fpqc}})$ of affine schemes over the base $\mathfrak{C}$ with the **fpqc** topology. Let h denote the canonical functor $A \longrightarrow A^\wedge$, $X \longmapsto A(-, X)$. To any presheaf of sets $\mathbf{X}$ on $(A, \mathfrak{T})$, we assign a fibered category $E_{\mathbf{X}}$ over the base $h/\mathfrak{F}$: the fiber over an object $(\mathbf{Spec}_{\mathfrak{C}}(R^\sim), \xi)$ of the category $h/\mathbf{X}$ is the forgetful functor from the category $R^\sim - mod_{\mathfrak{C}}$ of $R^\sim$-modules to the category C (i.e. an object of the category $RCat/C$). Thus we have a functorial map which assigns to any presheaf $\mathbf{X}$ of sets on the site $(A, \mathfrak{T})$ a 'space' $Qcoh_{\mathbf{X}/\mathfrak{C}}$ over C of quasi-coherent sheaves on $\mathfrak{F}$ and to any sheaf morphism $\mathbf{X} \longrightarrow \mathbf{Y}$ the corresponding morphism $Qcoh_{\mathbf{X}/\mathfrak{C}} \longrightarrow Qcoh_{\mathbf{Y}/\mathfrak{C}}$ of 'spaces' over $\mathfrak{C}$. We denote the restriction of this map to the category $Sh(A,\mathfrak{T})$ of sheaves of sets on $(A,\mathfrak{T})$ by $Qcoh_{\mathfrak{C}}$.

5.5.1. Proposition. *The functor $Qcoh_{\mathfrak{C}}$ has a canonical lifting to a functor $Qcoh_{\mathfrak{C}}^{\sim}$: $Sh(A,\mathfrak{T}) \longrightarrow Es_{\mathbf{fpqc}}/\mathfrak{C}$. The functor $Qcoh_{\mathfrak{C}}^{\sim}$ maps locally affine $\mathfrak{T}_{fpqc}^{\mathfrak{C}}$-spaces (resp. $\mathfrak{T}_{fpqc}^{\mathfrak{C}}$-schemes) to* **fpqc** *locally affine spaces (resp. schemes) over $\mathfrak{C}$.*

Proof. (a) To any sheaf $\mathbf{X}$ on $(A,\mathfrak{T})$, we assign canonically an object $(\mathbf{X},\mathfrak{t}_h)$ of the category $Es_{\mathfrak{T}|h}$ (cf. the part (c) of the argument of 4.7.1). To any sieve of $h/\mathbf{X}$ corresponds a sieve of $\mathbf{Spec}_{\mathfrak{C}}/Qcoh_{\mathbf{X}/\mathfrak{C}}$, and this correspondence maps equivalent sieves to equivalent sieves.

(b) Suppose $\mathbf{X}$ has a locally affine (resp. Zariski) $\mathfrak{T}$-cover. The functor $Qcoh_{\mathfrak{C}}^{\sim}$ maps this cover to an affine cover of $Qcoh_{\mathbf{X}/\mathfrak{C}}^{\sim}$ defining the structure of an **fpqc** locally affine space (resp. of a scheme) over the base $\mathfrak{C}$. Details are left to the reader. ∎

5.5.2. Proposition. *The category of* **fpqc** *locally affine spaces (resp. schemes) over $\mathfrak{C}$ is naturally equivalent to the category of locally affine $\mathfrak{T}_{\mathbf{fpqc}}$-spaces (resp. of $\mathfrak{T}_{\mathbf{fpqc}}$-schemes.*

Proof. By 4.7.1, for any subcanonical pair $(\mathfrak{T},F)$, the functor $h_{\mathfrak{T}|F} : (X,\mathfrak{t}) \longmapsto Es_{\mathfrak{T}|F}(F_{\mathfrak{T}}(-),(X,\mathfrak{t}))$ induces functors between the corresponding categories of locally affine spaces and schemes: $\mathfrak{Las}_{\mathfrak{T}|F} \longrightarrow \mathfrak{Las}_{\mathfrak{T}}$ and $\mathfrak{Sch}_{\mathfrak{T}|F} \longrightarrow \mathfrak{Sch}_{\mathfrak{T}}$. In the case when $(A,\mathfrak{T})$ is the site $(\mathbf{Aff}_{\mathcal{A}^{\sim}},\mathfrak{T}_{\mathbf{fpqc}}^{\mathfrak{C}})$, the functor $Qcoh^{\sim} : Sh(A,\mathfrak{T}) \longrightarrow Es_{\mathbf{fpqc}}/\mathfrak{C}$ of 5.5.1 induces the quasi-inverse functors. ∎

6. Examples of noncommutative schemes.

6.1. Vector fiber of an object. Let $\mathcal{A}^{\sim}$ be a monoidal category, and let $\mathcal{F}$ denote the forgetful functor

$$Alg\mathcal{A}^{\sim} \longrightarrow \mathcal{A}, \quad (R,\mu) \longmapsto R.$$

6.1.1. Lemma. *Let $E \in Ob\mathcal{A}$ be such that the functors $E \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$ and $- \otimes E$ preserve countable coproducts. And let there exists a coproduct $\oplus_{n\ge 0}E^{\otimes n}$. Then the functor $\mathcal{A}(E,\mathcal{F}-) : Alg\mathcal{A}^{\sim} \longrightarrow$* **Sets** *is corepresentable.*

Proof. Denote by $T(E)$ the algebra $(\oplus_{n\ge 0}E^{\otimes n},\mu_E)$, where the multiplication μ_E is given by the identical morphisms $E^{\otimes n} \otimes E^{\otimes m} \longrightarrow E^{\otimes(m+n)}$. For any algebra (R,μ) in $\mathcal{A}^{\sim}$, the natural map $\mathcal{A}(E,R) \longrightarrow Alg\mathcal{A}^{\sim}(T(E),(R,\mu))$ is a functorial isomorphism. ∎

6.1.2. Corollary. *Assume that $\mathcal{A}$ has countable coproducts and $\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ preserves countable coproducts in both arguments. Then the functor*

$$\mathcal{F} : Alg\mathcal{A}^{\sim} \longrightarrow \mathcal{A}, \quad (R,\mu) \longmapsto R$$

has a left adjoint.

6.1.3. Corollary. *Let $E_1, E_2 \in Ob\mathcal{A}$ be such that the functors $E_i \otimes -$ and $- \otimes E_i$, $i = 1,2$, preserve countable coproducts. Assume that $T(E_1)$, $T(E_2)$, and $T(E_1 \oplus E_2)$ exist. Then there is a natural algebra isomorphism $T(E_1 \oplus E_2) \simeq T(E_1) \star T(E_2)$.*

6.1.4. Vector fibers. Let $E \in Ob\mathcal{A}$ satisfy the conditions of 6.1.1. The *vector fiber,* $\mathbf{V}(E)$*, associated with* E is the affine scheme $\mathbf{Spec}_{\mathcal{A}^\sim}(T(E))$.

It follows that if the objects E_1, E_2 of $\mathcal{A}$ satisfy the assumptions of 6.1.3, then the product $\mathbf{V}(E_1) \times \mathbf{V}(E_2)$ exists and is isomorphic to $\mathbf{V}(E_1 \oplus E_2)$.

6.1.5. Proposition. *Let* $E, E' \in Ob\mathcal{A}$ *satisfy the assumptions of 6.1.1, and let* E' $\phi : E \longrightarrow E'$ *be a strict epimorphism. If* $\Phi(E')$ *is right exact, then the corresponding morphism* $\mathbf{V}(E') \longrightarrow \mathbf{V}(E)$ *is a closed immersion.*

Proof. Since $\phi : E \longrightarrow E'$ is a strict epimorphism and the functor $E \longmapsto T(E)$ is right exact (compatible with all colimits as any functor having a right adjoint), the corresponding algebra morphism $T(\phi) : T(E) \longrightarrow T(E')$ is a strict epimorphism, hence the assertion. ∎

6.2. Quasi-coherent modules on a vector fiber. Fix a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi)$. Let an object E of $\mathcal{A}$ is such that $\Phi^\sim(T(E))$ is naturally isomorphic to $T(\Phi(E))$ (which is the case if the functors $E \otimes -$ and Φ are compatible with countable coproducts). Then the category $Qcoh_{\mathbf{V}(E)/\mathfrak{C}}$ of quasi-coherent $\mathbf{V}(E)/\mathfrak{C}$-modules is isomorphic to the category of actions of $\Phi(E)$: its objects are pairs $(M, \Phi(M) \stackrel{\xi}{\longrightarrow} M)$, and morphisms and their composition are defined in an obvious way.

6.3. Vector fiber associated with an admissible pair of objects. Fix a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi)$.

6.3.1. Admissible pairs of objects. Finite objects. We call a pair (M, L) of objects of C $\mathfrak{C}$-*admissible* if the functor

$$\mathcal{A} \longrightarrow \mathbf{Sets}, \quad F \longmapsto C(M, \Phi(F)(L)) \tag{1}$$

is corepresentable, i.e. there is an object $L^\wedge M$ of $\mathcal{A}$ (defined uniquely up to isomorphism) and a functorial (in F) isomorphism $C(M, \Phi(F)(L)) \simeq \mathcal{A}(L^\wedge M, F)$. We call an object L of the category C *finite* if (M, L) is an admissible pair for any $M \in ObC$.

6.3.1.1. Example. Let $C = R - mod$ for an associative ring R, $\mathcal{A}^\sim$ is the monoidal category of R-bimodules. Then finite objects of the category C are projective R-modules of finite type. If L is a projective R-module of finite type and M an arbitrary left R-module, then $L^\wedge M \simeq M \otimes L^\vee$, where $L^\vee$ denotes the dual to L (right) R-module: $L^\vee \simeq R - mod(L, R)$. In particular, if $L = R$, then $L^\wedge M$ is isomorphic to the R-bimodule $M \otimes R_r$, where R_r is R regarded as a right R-module.

6.3.1.2. Example. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{A}^\sim, \Phi)$ be the left base corresponding to a monoidal category $\mathcal{A}^\sim$ (cf. 2.3.2.). Let $L, M \in Ob\mathcal{A}$. By definition, the pair (M, L) is admissible iff the functor $\mathcal{A} \longrightarrow \mathbf{Sets}$, $F \longmapsto C(M, F \otimes L))$ is corepresentable. Suppose that L is a *finite object*, i.e. there exists an object $L^!$ such that the functor $L^! \otimes -$ is a right adjoint to $L \otimes -$. Then $F \otimes L \simeq \mathbf{Hom}(L^!, F)$, hence

$$C(M, F \otimes L)) \simeq C(M, \mathbf{Hom}(L^!, F)) \simeq C(M \otimes L^!, F)$$

functorially in F. In other words, L is finite object of the monoidal category $\mathcal{A}^\sim$ iff it is a finite object (in the sense of 6.3.1) of the left base.

6.3.2. Proposition. *Let $(\Psi^\sim, g^*; \lambda) : \mathfrak{C}' = (\mathcal{A}'^\sim, C', \Phi'^\sim) \longrightarrow \mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ be a morphism (cf. 3.6) such that the functor Ψ has a left adjoint, Ψ^*. Then for any $\mathfrak{C}$-admissible pair (L, M), the pair $(g^*(L), g^*(M))$ is $\mathfrak{C}'$-admissible.*

Proof. In fact, we have functorial isomorphisms

$$\mathcal{A}'(\Psi^*(L^\wedge M), E') \simeq \mathcal{A}(L^\wedge M, \Psi(E')) \simeq$$

$$C(M, \Phi\Psi(E')(L)) \simeq C'(g^*(M), \Phi'(E')(g^*(L)))$$

Hence the assertion. ∎

6.3.3. Corollary. *Let $\mathcal{A}^\sim$ (resp. $\mathcal{A}'^\sim$) be the category of continuous (i.e. having a right adjoint) endofunctors of a category C (resp. C'). Let $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$ and $\mathfrak{C}' = (\mathcal{A}'^\sim, C', \Phi'^\sim)$ be the corresponding actions, and let $g : C' \longrightarrow C$ be a continuous morphism such that g_* has a right adjoint. Then for any $\mathfrak{C}$-admissible pair (L, M), the pair $(g^*(L), g^*(M))$ is $\mathfrak{C}'$-admissible. In particular, $(g^*(L), g^*(M))$ is admissible for any affine morphism g.*

Proof. 1) The morphism g induces a morphism $(\Psi_g^\sim, g) : \mathfrak{C}' = (\mathcal{A}'^\sim, C', \Phi'^\sim) \longrightarrow \mathfrak{C} = (\mathcal{A}^\sim, C, \Phi^\sim)$, and the functor Ψ_g maps every continuous functor $F' : C' \longrightarrow C'$ (– an object of $\mathcal{A}'$) to $g_*F'g^*$. It has a left adjoint, Ψ_g^*, which maps any continuous functor $F : C \longrightarrow C$ to g^*Fg_*. By 6.3.2, the pair $(g^*(L), g^*(M))$ is admissible and (by the argumen of 6.3.2) $(g^*(L))^\wedge(g^*(M)) = \Psi_g^*(L^\wedge M) = g^*(L^\wedge M)g_*$.

2) One can give an independent (on 6.3.2) argument. For any continuous functor $F : C \longrightarrow C$, the functor g_*Fg^* is continuous, and we have canonical isomorphisms

$$C'(g^*(L), Fg^*(M)) \simeq C(L, g_*Fg^*(M)) \simeq \mathcal{A}(L^\wedge M, g_*Fg^*(M)) \simeq \mathcal{A}'(g^*(L^\wedge M)g_*, F)$$

Since these isomorphisms are functorial in F, it follows that the pair $(g^*(L), g^*(M))$ is admissible and $g^*(L)^\wedge g^*(M) \simeq g^*(L^\wedge M)g_*$. ∎

6.3.4. Vector fibers. With any $\mathfrak{C}$-admissible pair (M, L) of objects of C, we associate a vector fiber $\mathbf{V}(L^\wedge M)$.

6.3.4.1. Proposition. *Let (L, M) and (L, M') be admissible pairs of objects.*

(a) The pair $(L, M \oplus M')$ is admissible and $\mathbf{V}(L^\wedge(M \oplus M')) \simeq \mathbf{V}(L^\wedge M) \times \mathbf{V}(L^\wedge M')$.

(b) Let $\phi : M \longrightarrow M'$ be a strict epimorphism. Then the corresponding morphism $\mathbf{V}(L^\wedge M') \longrightarrow \mathbf{V}(L^\wedge M)$ is a closed immersion.

Proof. (a) We have

$$C(M \oplus M', \Phi(-)(L)) \simeq C(M, \Phi(-)(L)) \prod C(M', \Phi(-)(L)) \simeq$$

$$\mathcal{A}(L^\wedge M, -) \prod \mathcal{A}(L^\wedge M', -) \simeq \mathcal{A}(L^\wedge M \oplus L^\wedge M', -)$$

which shows that the pair $(L, M \oplus M')$ is admissible and $L^\wedge(M \oplus M') \simeq (L^\wedge M) \oplus (L^\wedge M')$. The assertion follows now from 6.1.4.

(b) The assertion follows from 6.1.5. Details are left to the reader. ∎

6.3.5. Interpretation. For any affine $\mathcal{A}^\sim$-scheme $\mathbf{X} = \mathbf{Spec}_{\mathcal{A}^\sim}(R)$, denote by $f_{\mathbf{X}}^*$ the inverse image functor $L \longmapsto \Phi^\sim(R)(L)$ of the canonical morphism $f_{\mathbf{X}} : Qcoh_{\mathbf{X}/\mathfrak{C}} \longrightarrow C$.

For any two objects M, L of the category C, consider the functor

$$\mathcal{H}_{M,L} : \mathbf{Aff}_{\mathcal{A}^\sim}^{op} \longrightarrow \mathbf{Sets}, \quad \mathbf{X} \longmapsto Qcoh_{\mathbf{X}/\mathfrak{C}}(f_{\mathbf{X}}^*(M), f_{\mathbf{X}}^*(L)) \tag{1}$$

6.3.5.1. Proposition. *If the pair (M, L) is admissible, then the functor $\mathcal{H}_{M,L}$ is representable by the affine scheme $\mathbf{V}(L^\wedge M)$.*

Proof. In fact, for any $\mathbf{X} = \mathbf{Spec}(R^\sim)$, we have the following functorial isomorphisms:

$$\mathcal{H}_{M,L}(\mathbf{X}) := Qcoh_{\mathbf{X}/\mathfrak{C}}(f_{\mathbf{X}}^*(M), f_{\mathbf{X}}^*(L)) \simeq C(M, \Phi(R)(L)) \simeq$$

$$\mathcal{A}(L^\wedge M, R) \simeq Alg_{\mathcal{A}^\sim}(T(L^\wedge M), R^\sim) = \mathbf{Aff}_{\mathcal{A}^\sim}(\mathbf{X}, \mathbf{V}(L^\wedge M)) \tag{1}$$

Hence the assertion. ∎

6.4. The group scheme GL_V. Fix objects V, W of the category C. We have a functor

$$Iso_{V,W} : \mathbf{Aff}_{\mathcal{A}^\sim}^{op} \longrightarrow \mathbf{Sets}, \quad \mathbf{X} \longmapsto Aut_{Qcoh_{\mathbf{X}/\mathfrak{C}}}(f_{\mathbf{X}}^*(V), f_{\mathbf{X}}^*(W)) \tag{1}$$

6.4.1. Proposition. *Let V, W be objects of C such that the pairs (V, W) and (W, V) are admissible. Then the functor $Iso_{V,W}$ is representable by an affine $\mathcal{A}^\sim$-scheme.*

Proof. (a) Let (V, W) and (W, Z) be admissible pairs of objects of C. Consider the functor $G_{V,W,Z} : \mathbf{Aff}_{\mathcal{A}^\sim}^{op} \longrightarrow \mathbf{Sets}$ which assigns to any affine scheme $\mathbf{X} = \mathbf{Spec}_{\mathcal{A}^\sim}(R)$ the pair of morphisms $f_{\mathbf{X}}^*(V) \xrightarrow{v} f_{\mathbf{X}}^*(W) \xrightarrow{w} f_{\mathbf{X}}^*(Z)$. The functor $G_{V,W,Z}$ is representable by $\mathbf{V}(W^\wedge V) \times \mathbf{V}(Z^\wedge W)$.

(b) Let (V, W), (W, V), and (V, V) be admissible pairs of objects of C. Denote by $\Phi_{V,W}$ the subfunctor of the functor $G = G_{V,W,V}$ which assigns to any affine scheme $\mathbf{X}$ the subset of all pairs $(v, u) \in G_{V,W,V}(\mathbf{X})$ such that $u \circ v = id$. We claim that the functor $\Phi_{V,W}$ is representable by an affine scheme.

Consider two functorial maps

$$\alpha_{\mathbf{X}}, \beta_{\mathbf{X}} : G_{V,W,V}(\mathbf{X}) \longrightarrow Qcoh_{\mathbf{X}/\mathfrak{C}}(f_{\mathbf{X}}^*(V), f_{\mathbf{X}}^*(V))$$

defined by $\alpha_{\mathbf{X}}(v, u) = u \circ v$ and $\beta_{\mathbf{X}}(v, u) = id_{f_{\mathbf{X}}^*(V)}$. Since the pair (V, V) is admissible,

$$Qcoh_{\mathbf{X}/\mathfrak{C}}(f_{\mathbf{X}}^*(V), f_{\mathbf{X}}^*(V)) \simeq \mathbf{Aff}_{\mathcal{A}^\sim}(\mathbf{X}, \mathbf{V}(V^\wedge V))$$

Let α', β' denote the corresponding morphisms

$$\mathbf{V}(W^\wedge V \oplus V^\wedge W) \simeq \mathbf{V}(W^\wedge V) \times \mathbf{V}(V^\wedge W) \longrightarrow \mathbf{V}(V^\wedge V)$$

The functor $\Phi_{V,W}$ is representable by the kernel of the pair of morphisms α', β'.

(c) Let (V, W), (W, V), (V, V), and (W, W) be admissible pairs of objects of C. And let $\mathbf{X}$ be an affine $\mathcal{A}^\sim$-scheme. The set $Iso(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(W))$ is naturally isomorphic to the set of pairs (u, v), where $u : f^*_{\mathbf{X}}(V) \longrightarrow f^*_{\mathbf{X}}(W)$ and $v : f^*_{\mathbf{X}}(W) \longrightarrow f^*_{\mathbf{X}}(V)$ are quasi-coherent module morphisms such that $u \circ v = id$ and $v \circ u = id$. So $Iso(f^*_{\mathbf{X}}(V), f^*_{\mathbf{X}}(W))$ is identified with the fiber product of the morphisms

$$\Phi_{V,W}(\mathbf{X}) \xrightarrow{\phi} G_{V,W,V}(\mathbf{X}) \xleftarrow{\psi} \Phi_{W,V}(\mathbf{X})$$

where $\phi = \phi_{V,W}$ is the natural embedding, ψ is the composition of the natural embedding $\phi_{W,V} : \Phi_{W,V} \longrightarrow G_{W,V,W}$ and the functorial isomorphism

$$G_{W,V,W} \longrightarrow G_{V,W,V}, \quad (u, v) \longmapsto (v, u)$$

Thus the map $Iso_{V,W}$ is extended to the functor which is the kernel of the pair of functor morphisms (ϕ, ψ). Since both source and target of the arrows ϕ, ψ, the functors $\Phi_{V,W}$ and $G_{V,W,V}$, are representable, the functor $Iso_{V,W}$ is representable too. ∎

6.4.2. Corollary. *Let V be an object of C such that the pair (V, V) is admissible. The functor Aut_V is representable by an affine $\mathcal{A}^\sim$-scheme in groups.*

6.5. Grassmannians. Fix a pair (L, M) of objects of the category C. For any affine $\mathcal{A}^\sim$-scheme $\mathbf{X} = \mathbf{Spec}(R)$, denote by $Grass_{L,M}(\mathbf{X})$ the set of all isomorphy classes of morphisms $v : f^*_{\mathbf{X}}(M) \longrightarrow L'$ which are *locally split over* C in the following sense: there exists an **lqc** cover $\mathfrak{U}$ of $\mathbf{X}$ and for any $(U \xrightarrow{u} \mathbf{X}) \in \mathfrak{U}$, a morphism $\phi_u : L \longrightarrow M$ such that the composition $u^*(v) \circ u^* f^*_{\mathbf{X}}(\phi_u) : u^* f^*_{\mathbf{X}}(L) \longrightarrow u^*(L')$ is an isomorphism.

6.5.1. Proposition. *The map $\mathbf{X} \longmapsto Grass_{L,M}(\mathbf{X})$ is naturally extended to a functor $(\mathbf{Aff}_{\mathcal{A}^\sim})^{op} \longrightarrow \mathbf{Sets}$.*

Proof. For any morphism $t : \mathbf{Y} \longrightarrow \mathbf{X}$ and any element $[v]$ of $Grass_{L,M}(\mathbf{X})$, $v : f^*_{\mathbf{X}}(M) \longrightarrow L'$, the equivalence class of the morphism $t^*(v) : t^* f^*_{\mathbf{X}}(M) \longrightarrow t^*(L')$ belongs to $Grass_{L,M}(\mathbf{Y})$.

In fact, by definition of $Grass_{L,M}(X, f)$, there exists a cover $\mathfrak{U}$ of $\mathbf{X}$ such that for any $U = (U \xrightarrow{u} \mathbf{X}) \in \mathfrak{U}$, there is a morphism $\phi_U : L \longrightarrow M$ for which $u^*(v) \circ u^* f^*_{\mathbf{X}}(\phi_U)$ is an isomorphism. This implies that $t^*(v)$ splits in the desired way over the cover $t^*\mathfrak{U} = \{U \times_{\mathbf{X}} \mathbf{Y} \longrightarrow \mathbf{Y} \mid U \in \mathfrak{U}\}$. ∎

6.5.2. Theorem. *Let (L, M) and (L, L) be admissible pairs of objects of the category C. Assume that $\mathcal{A}$ has countable coproducts and for any $F \in Ob\mathcal{A}$, the functor $F \otimes -$, preserves them. Then the functor $Grass_{L,M} : (\mathbf{Aff}_{\mathcal{A}^\sim})^{op} \longrightarrow \mathbf{Sets}$ is representable by a scheme.*

Proof. It follows from the definition that the functor $Grass_{L,M}$ induces a sheaf of sets on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes over $\mathfrak{C}$ which we also denote by $Grass_{L,M}$. It remains to show the existence of an affine cover defining the structure of a scheme on $Grass_{L,M}$. Actually the affine cover constructed below is semiseparated.

Fix a morphism $\phi : L \longrightarrow M$. For any $\mathbf{X} \in Ob(\mathbf{Aff}_{\mathcal{A}^\sim})$, consider the subset $F_{\phi;L,M}(\mathbf{X})$ of $Grass_{L,M}(\mathbf{X})$ formed by the isomorphy classes of all morphisms $v : f^*_{\mathbf{X}}(M) \longrightarrow L'$ such that $v \circ f^*_{\mathbf{X}}(\phi)$ is an isomorphism.

6.5.2.1. Lemma. *The map* $\mathbf{X} \longmapsto F_{\phi;L,M}(\mathbf{X})$ *is naturally extended to a functor* $F_{\phi;L,M} : (\mathbf{Aff}_{\mathcal{A}^\sim})^{op} \longrightarrow \mathbf{Sets}$ *which is a subfunctor of* $Grass_{L,M}$.

The functor $F_{\phi;L,M}$ *is representable by an affine scheme.*

Proof. (i) In fact, if $v : f^*_{\mathbf{X}}(M) \longrightarrow L'$ belongs to $F_{\phi;L,M}(\mathbf{X})$, i.e. $v \circ f^*_{\mathbf{X}}(\phi)$ is an isomorphism, then for any morphism $h : \mathbf{Y} \longrightarrow \mathbf{X}$, the composition $h^*(v) \circ h^* f^*_{\mathbf{X}}(\phi)$ is an isomorphism. Clearly $F_{\phi;L,M}$ is a subfunctor of $Grass_{L,M}$.

(ii) Note that one can identify $F_{\phi;L,M}(\mathbf{X})$ with the set of epimorphisms $v : f^*(M) \longrightarrow f^*(L)$ such that $v \circ f^*(\phi) = id_{f^*(L)}$. In fact, if $v' : f^*(M) \longrightarrow L'$ is such that $w := v' \circ f^*(\phi) : f^*(L) \longrightarrow L'$ is an isomorphism, then $v = w^{-1} \circ v'$ has the required property.

(iii) There are two maps

$$\alpha_{\mathbf{X}}, \beta_{\mathbf{X}} : X(f^*_{\mathbf{X}}(M), f^*_{\mathbf{X}}(L)) \rightrightarrows C(f^*(L), f^*(L))$$

defined by $\alpha_{\mathbf{X}} : v \longmapsto v \circ f^*_{\mathbf{X}}(\phi)$, $\quad \beta_{\mathbf{X}} : v \longmapsto id_{f^*_{\mathbf{X}}(L)}$. The maps $\alpha_{\mathbf{X}}$ and $\beta_{\mathbf{X}}$ are functorial in $\mathbf{X}$, hence they define morphisms resp. α and β from the functor $\mathbf{X} \longmapsto X(f^*_{\mathbf{X}}(M), f^*_{\mathbf{X}}(L))$ to the functor $\mathbf{X} \longmapsto X(f^*_{\mathbf{X}}(L), f^*_{\mathbf{X}}(L))$. The first functor is representable by $\mathbf{V}(L^\wedge M)$ and the second one is representable by $\mathbf{V}(L^\wedge L)$. Let α' and β' be morphisms from $\mathbf{V}(L^\wedge M)$ to $\mathbf{V}(L^\wedge L)$ corresponding to resp. α and β. The functor $F_{\phi;L,M} : X \longmapsto F_{\phi;L,M}(X)$ is the kernel of the pair (α, β), hence it is representable by the kernel, $F_{\phi;L,M}$, of the pair (α', β') of affine scheme morphisms. ■

6.5.2.2. Each of the functor morphisms $F_{\phi;L,M} \longrightarrow Grass_{L,M}$ is representable; i.e. for any affine $\mathcal{A}^\sim$-scheme $\mathbf{X}$ and any morphism $h_{\mathbf{X}} \longrightarrow Grass_{L,M}$, the functor

$$\mathbf{Y} \longmapsto F_{\phi;L,M}(\mathbf{Y}) \times_{Grass_{L,M}(\mathbf{Y})} h_{\mathbf{X}}(\mathbf{Y})$$

is representable by an affine subscheme of $\mathbf{X}$.

In fact, any morphism $h_{\mathbf{X}} \longrightarrow Grass_{L,M}$ is uniquely determined by an element of $Grass_{L,M}(\mathbf{X})$, i.e. by the equivalence class $[v]$ of a locally split epimorphism $v : f^*_{\mathbf{X}}(M) \longrightarrow f^*_{\mathbf{X}}(L)$. The corresponding map $h_{\mathbf{X}}(\mathbf{Y}) \longrightarrow Grass_{L,M}(\mathbf{Y})$ sends any morphism $t : \mathbf{Y} \longrightarrow \mathbf{X}$ into $[t^*(v)]$. The fiber product $F_{\phi;L,M}(\mathbf{Y}) \times_{Grass_{L,M}(\mathbf{Y})} h_{\mathbf{X}}(\mathbf{Y})$ consists of all pairs (w, t), where $t \in h_{\mathbf{X}}(\mathbf{Y})$ and $[w : g^*(M) \longrightarrow g^*(L)]$ are such that $w \circ f^*_{\mathbf{X}}(\phi) = id_{f^*_{\mathbf{X}}(L)}$ and $w = t^*(v)$. Since v and ϕ here are fixed, the fiber product $F_{\phi;L,M}(\mathbf{Y}) \times_{Grass_{L,M}(\mathbf{Y})} h_{\mathbf{X}}(\mathbf{Y})$ can be identified with the set of all morphisms $t : \mathbf{Y} \longrightarrow \mathbf{X}$ such that $t^*(v \circ f^*_{\mathbf{X}}(\phi)) = id_{t^* f^*_{\mathbf{X}}(L)}$. In other words, the fiber product $F_{\phi;L,M}(\mathbf{Y}) \times_{Grass_{L,M}(\mathbf{Y})} h_{\mathbf{X}}(\mathbf{Y})$ is identified with the kernel of the pair of morphisms $\beta_{\mathbf{Y}}, \alpha_{\mathbf{Y}} : h_{\mathbf{X}}(\mathbf{Y}) \longrightarrow Qcoh_{\mathbf{Y}/\mathfrak{C}}(f^*_{\mathbf{Y}}(L), f^*_{\mathbf{Y}}(L))$, defined by $\beta_{\mathbf{Y}} : t \longmapsto id_{f^*_{\mathbf{Y}}(L)}$, $\alpha_{\mathbf{Y}} : t \longmapsto t^*(v \circ f^*_{\mathbf{X}}(\phi))$. The morphisms $\beta_{\mathbf{Y}}$, $\alpha_{\mathbf{Y}}$ are functorial in $\mathbf{Y}$, and $Qcoh_{\mathbf{Y}/\mathfrak{C}}(f^*_{\mathbf{Y}}(L), f^*_{\mathbf{Y}}(L)) \simeq h_{\mathbf{V}(L^\wedge L)}(\mathbf{Y})$. Hence the morphisms $\beta = (\beta_{\mathbf{Y}})$, $\alpha = (\alpha_{\mathbf{Y}})$ define a pair of morphisms $\alpha', \beta' : \mathbf{X} \rightrightarrows \mathbf{V}(L^\wedge L)$, and the functor $\mathbf{Y} \longmapsto F_{\phi;L,M}(\mathbf{Y}) \times_{Grass_{L,M}(\mathbf{Y})} h_{\mathbf{X}}(\mathbf{Y})$ is representable by the kernel of the pair (α', β').

6.5.2.3. The functor morphisms $F_{\phi;L,M} \longrightarrow Grass_{L,M}$ form a cover of $Grass_{L,M}$ in the topology $\mathfrak{T}_{\mathbf{lqc}}$.

It suffices to show that for any affine scheme $\mathbf{X}$ and any morphism $\psi : h_{\mathbf{X}} \longrightarrow Grass_{L,M}$, the canonical morphisms $F_{\phi;L,M} \times_{Grass_{L,M}} \mathbf{X} \longrightarrow \mathbf{X}$ form an affine cover of $\mathbf{X}$. In fact, the morphism ψ corresponds to an element $[f_{\mathbf{X}}^*(M) \xrightarrow{v} L']$ of the set $Grass_{L,M}(\mathbf{X})$. By the definition of the Grassmannian, there exists a cover $\mathfrak{U}$ of $\mathbf{X}$ and for any $(U \xrightarrow{u} \mathbf{X}) \in \mathfrak{U}$, a morphism $\phi_u : L \longrightarrow M$ such that the composition $u^*(v) \circ u^* f_{\mathbf{X}}^*(\phi_u) : u^* f_{\mathbf{X}}^*(L) \longrightarrow u^*(L')$ is an isomorphism. Thus, for any $(U \xrightarrow{u} \mathbf{X}) \in \mathfrak{U}$, we have a commutative diagram

$$\begin{array}{ccc} h_U & \xrightarrow{\phi_u} & F_{\phi_u;L,M} \\ \downarrow & & \downarrow \\ h_{\mathbf{X}} & \xrightarrow{\psi} & Grass_{L,M} \end{array} \qquad (1)$$

Here the morphism ϕ_u corresponds to the element $[u^*(v)] \in F_{\phi_u;L,M}(U)$.

6.5.2.4. It follows from the argument of 6.5.2.2 that the cover

$$\{F_{\phi;L,M} \longrightarrow Grass_{L,M} |\ \phi : L \longrightarrow M\}$$

is semiseparated. In particular, it defines a structure of a semiseparated scheme on the sheaf $Grass_{L,M}$. ■ ■

6.5.3. Fundamental element. The $\mathfrak{C}$-scheme which represents the functor

$$\mathbf{X} \longmapsto Grass_{L,M}(\mathbf{X})$$

will be called *L-th Grassmannian of M* and denoted by $\mathbf{Grass}_{L,M} := (Qcoh_{Grass_{L,M}/\mathfrak{C}}, \pi)$. Here π is the morphism $Qcoh_{Grass_{L,M}/\mathfrak{C}} \longrightarrow C$. There is a canonical element $[\mathfrak{v}] = [\pi^*(M) \xrightarrow{\mathfrak{v}} L(1)]$ such that the map

$$h_{\mathbf{Grass}_{L,M}}(\mathbf{X}) \longrightarrow Grass_{L,M}(\mathbf{X}), \quad t \longmapsto t^*(\mathfrak{v})$$

is a functor isomorphism. We call $[\mathfrak{v}]$ *the fundamental element on* $\mathbf{Grass}_{L,M}$.

6.5.4. Projective fibers. Suppose L is such a finite object that $L^\wedge M \simeq M$ for any $M \in ObC$. Then we will write $\mathbf{P}(M)$ instead of $\mathbf{Grass}_{L,M}$ and call $\mathbf{P}(M)$ *the projective fiber over C defined by M*. In this case we shall write the fundamental element as $[\pi^*(M) \xrightarrow{\mathfrak{v}} \mathcal{O}(1)]$.

6.5.4.1. Example. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{A}^\sim, \mathfrak{L}^\sim)$ be the left base. Then $\mathbf{P}(M) = \mathbf{Grass}_{\mathbf{1},M}$, where $\mathbf{1}$ is the unit object of the monoidal category $\mathcal{A}^\sim$.

6.5.5. Functoriality. The map $M \longmapsto Grass_{L,M}$ is not functorial with respect to M. It is functorial, however, with respect to the class $\mathfrak{Epi}_H$ of *L-epimorphisms*. The latter are defined as morphisms $\psi : M' \longrightarrow M$ such that the corresponding map $C(L, \psi) : C(L, M') \longrightarrow C(L, M)$ is surjective.

If ψ is a *coretraction*, i.e. there exists $\phi : M \longrightarrow M'$ such that $\psi \circ \phi = id_M$, then ψ is an L-epimorphism for any object L of the category C. If L is a projective object, then any epimorphism is an L-epimorphism.

6.5.5.1. Proposition. *Let $L \in ObC$. To any L-epimorphism $\psi : M' \longrightarrow M$, there corresponds a functor morphism $Grass_L(\psi) : Grass_{L,M} \longrightarrow Grass_{L,M'}$.*

The morphism $Grass_{L,M}(\psi) : Grass_{L,M'} \longrightarrow Grass_{L,M}$ is a closed immersion.

Proof. (a) Let $\mathbf{X}$ be a $\mathfrak{C}$-scheme and $v : f_{\mathbf{X}}^*(M) \longrightarrow L'$ a locally split morphism; i.e. there exists a cover $\mathfrak{U}$ of $\mathbf{X}$ and for any $(U \xrightarrow{u} X) \in \mathfrak{U}$, a morphism $\phi_u : L \longrightarrow M$ such that $u^*(v) \circ u^* f_{\mathbf{X}}^*(\phi_u)$ is an isomorphism. Since $\psi : M' \longrightarrow M$ is an L-epimorphism, morphisms ϕ_u are 'lifted' to M'; i.e. there exists $\phi'_u : L \longrightarrow M'$ such that $\phi_u = \psi \circ \phi'_u$. This gives a functorial map $Grass_L(\psi) : Grass_{L,M}(\mathbf{X}) \longrightarrow Grass_{L,M'}(\mathbf{X})$.

(b) The argument is similar to that of [GrD], 9.7.9. ∎

6.6. Flag varieties.

6.6.1. Fix a category C and an object M of C. Let $\mathbf{L} = (L_i)_{1 \le i \le p}$ be an increasing sequence of objects of C such that the pairs (M, L_i) and (L_i, L_i) are admissible for all i. For any $\mathbf{X} \in ObRCat_{\mathfrak{C}}$, denote by $Fl_{\mathbf{L},M}(\mathbf{X})$ the subset of $\prod_{1 \le i \le p} Grass_{L_i,M}(\mathbf{X})$ formed by $([v_1], ..., [v_p])$, $v_i : f_{\mathbf{X}}^*(M) \longrightarrow f_{\mathbf{X}}^*(L_i)$, such that $Ker(v_{i+1}) \subseteq Ker(v_i)$, $1 \le i \le p-1$. Elements of $Fl_{\mathbf{L},M}(\mathbf{X})$ will be called *flags of the type* $\mathbf{L}$ *in* M.

6.6.2. Lemma. *The map $\mathbf{X} \longmapsto Fl_{\mathbf{L},M}(\mathbf{X})$ is naturally extended to a contravariant functor, $Fl_{\mathbf{L},M} : \mathbf{Aff}_{\mathcal{A}^\sim}^{op} \longrightarrow \mathbf{Sets}$.*

Proof. The extension of the map $\mathbf{X} \longmapsto Fl_{\mathbf{L},M}(\mathbf{X})$ to morphisms is uniquely determined by the condition that the canonical embeddings $Fl_{\mathbf{L},M}(\mathbf{X}) \longrightarrow \prod_{1 \le i \le p} Grass_{L_i,M}(\mathbf{X})$ define a functorial morphism

$$i_{\mathbf{L}} = i_{\mathbf{L},M} : Fl_{\mathbf{L},M} \longrightarrow \prod_{1 \le i \le p} Grass_{L_i,M}$$

Details are left to the reader. ∎

6.6.3. Proposition. *The functor $Fl_{\mathbf{L},M} : \mathbf{X} \longmapsto Fl_{\mathbf{L},M}(\mathbf{X})$ is a scheme and the morphism*

$$\mathbf{i}_{\mathbf{L},M} : Fl_{\mathbf{L},M} \longrightarrow \prod_{1 \le i \le p} Grass_{L_i,M} \tag{2}$$

is a closed immersion.

Proof is left to the reader. ∎

6.6.4. Let $\mathbf{H} = (H_i | 1 \le i \le p)$ be a set of objects of the category C. We call a morphism $\psi : M \longrightarrow M'$ *an* $\mathbf{H}$*-epimorphism* if it is an H_i-epimorphism for all i (cf. 6.5.5).

6.6.5. Proposition. *Fix objects M, M' of the category C. Let $\mathbf{L} = (L_i)_{1 \le i \le p}$ be an increasing sequence of objects of C such that the pairs (M, L_i), (M', L_i), and (L_i, L_i)*

are admissible for all i. To any $\mathbf{L}$-epimorphism $\psi : M \longrightarrow M'$, there corresponds a closed immersion $Fl_{\mathbf{H}}(\psi) : Fl_{\mathbf{L},M'} \longrightarrow Fl_{\mathbf{L},M}$.

Proof. In fact, there is a commutative diagram

$$\begin{array}{ccc} Fl_{\mathbf{L},M'}(\mathbf{X}) & \xrightarrow{i_{M'}(X)} & \prod_{1\le i\le p} Grass_{L_i,M'}(\mathbf{X}) \\ i_v(X)\downarrow & & \downarrow \\ Fl_{\mathbf{L},M}(\mathbf{X}) & \xrightarrow{i_M(X)} & \prod_{1\le i\le p} Grass_{L_i,M}(\mathbf{X}) \end{array}$$

functorial in $\mathbf{X}$, which induces a commutative diagram

$$\begin{array}{ccc} Fl_{\mathbf{L},M'} & \xrightarrow{i_{M'}} & \prod_{1\le i\le p} Grass_{L_i,E'} \\ Fl_{\mathbf{L}}(v)\downarrow & & \downarrow \prod_{1\le i\le p} Grass_{L_i,v} \\ Fl_{\mathbf{L},M} & \xrightarrow{i_M} & \prod_{1\le i\le p} Grass_{L_i,v} \end{array}$$

Since i_M, $i_{M'}$ and $Grass_{L_i,v}$ are closed immersions, $Fl_{\mathbf{L},v}$ is a closed immersion. ■

6.6.6. Let $\mathbf{L}' = (L_{i_k})_{1\le k\le q}$ be an increasing subsequence of $\mathbf{L}$, where $(i_k)_{1\le k\le q}$ is a strictly increasing sequence of integers in the interval $[1,p]$. There is a canonical morphism

$$p_M^{\mathbf{L}',\mathbf{L}}(X) : Fl_{\mathbf{L},M}(\mathbf{X}) \longrightarrow Fl_{\mathbf{L}',M}(\mathbf{X})$$

which maps any flag $([v_i])$ of type $\mathbf{L}$ into the flag $([v_{i_k}])$ of type $\mathbf{L}'$, hence a canonical morphism

$$p_M^{\mathbf{L}',\mathbf{L}} : Fl_{\mathbf{L},M} \longrightarrow Fl_{\mathbf{L}',M}$$

There is a commutative diagram

$$\begin{array}{ccc} Fl_{\mathbf{L},M}(\mathbf{X}) & \xrightarrow{i_{M'}(X)} & \prod_{1\le i\le p} Grass_{L_i,M}(\mathbf{X}) \\ i_v(X)\downarrow & & \downarrow \\ Fl_{\mathbf{L}',M}(\mathbf{X}) & \xrightarrow{i_M(X)} & \prod_{1\le k\le q} Grass_{L_{i_k},M}(\mathbf{X}) \end{array}$$

functorial in $\mathbf{X}$, which induces a commutative diagram

$$\begin{array}{ccc} Fl_{\mathbf{L},M} & \xrightarrow{i_{E'}} & \prod_{1\le i\le p} Grass_{L_i,E'} \\ Fl_{\mathbf{L},v}\downarrow & & \downarrow \prod_{1\le i\le p} Grass_{L_i,v} \\ Fl_{\mathbf{L}',M} & \xrightarrow{i_M} & \prod_{1\le k\le q} Grass_{L_{i_k},v} \end{array}$$

where the right vertical arrow is the canonical projection.

The following examples are studied in a bigger detail in [R3] and the last one, quantum flag variety, plays a central role in [LR3]. We reproduce them here to make a picture more complete.

6.7. Projective spectrum and a quasi-affine space related to a graded algebra. Let k be a commutative ring, Γ a commutative directly ordered group with the unit element 0. And let R be an associative Γ-graded k-algebra.

6.7.1. The cone of R. For any $\gamma \in \Gamma$, set $R_{>\gamma} := \oplus_{\sigma>\gamma} R_\sigma$. For any R-module M and any $\gamma \in \Gamma$, denote by M_γ the subset of all elements of M annihilated by $R_{>\gamma}$. Denote by $\mathcal{T}_+$ the full subcategory of the category $R-mod$ generated by all R-modules M such that $M = \sup\{M_\gamma | \gamma \in \Gamma\}$. One can see that $\mathcal{T}_+$ is a Serre subcategory of the category $R-mod$. Thus we have the quotient category $Cone_\Gamma(R) := R-mod/\mathcal{T}_+$ and a canonical continuous morphism (localization) $R-mod/\mathcal{T}_+ \longrightarrow R-mod$. Composing it with the canonical morphism (global section functor) $R-mod \longrightarrow R_0-mod$, we obtain an R_0-'space' $\mathbf{Cone}_\Gamma(R) := (R-mod/\mathcal{T}_+, \pi)$ called *the quasi-affine space* (or *affine cone*) *of the algebra R.*

6.7.2. Projective spectrum of R. Let $\mathcal{F}$ be the natural functor from the category $\mathfrak{gr}_\Gamma R-mod$ of Γ-graded R-modules to the category $R-mod$. And let $\mathfrak{T}_+$ denote the preimage of the Serre subcategory $\mathcal{T}_+$ in $\mathfrak{gr}_\Gamma R-mod$. Since the functor $\mathcal{F}$ is exact, $\mathfrak{T}_+$ is a Serre subcategory of $\mathfrak{gr}_\Gamma R-mod$, hence we have a quotient category $Proj_\Gamma(R) = \mathfrak{gr}_\Gamma R-mod/\mathfrak{T}_+$ and the following canonical continuous morphisms:

The morphism $\pi' : \mathbf{Proj}_\Gamma(R) \longrightarrow \mathfrak{gr}R-mod$ having a localization at $\mathfrak{T}_+$ as an inverse image functor.

The morphism $\varphi : \mathfrak{gr}_\Gamma R-mod \longrightarrow R_0-mod$, with the direct image functor

$$\varphi_* : \mathfrak{gr}_\Gamma R-mod \longrightarrow R_0-mod, \quad M = \oplus_{\gamma\in\Gamma} M_\gamma \longmapsto M_0. \tag{1}$$

Note by passing that the inverse image functor $\varphi^* : V \longmapsto R \otimes_{R_0} V$ is fully faithful. This follows from the fact that the adjunction morphism

$$\eta : Id_{R_0-mod} \longrightarrow \varphi_* \circ \varphi^*, \quad \eta(V) : V \longrightarrow (R \otimes_{R_0} V)_0 = R_0 \otimes_{R_0} V$$

is an isomorphism. Set $\pi^\Gamma = \varphi \circ \pi' : Proj_\Gamma(R) \longrightarrow R_0-mod$. We call $\mathbf{Proj}_\Gamma(R) := (Proj_\Gamma(R), \pi^\Gamma)$ the *projective spectrum of R.* The direct image of π^Γ is regarded as *the global sections functor.*

6.8. Affine covers. Let Γ and R be as in Subsection 6.7. Fix a Γ-graded associative k-algebra R. If $R_{>0}$ is an ideal of R (for instance, $\Gamma = \mathbb{Z}$ and $R_n = 0$ if $n < 0$), then the canonical morphism $Cone_\Gamma \longrightarrow R-mod$ can be regarded as the complement to the closed subspace of $R-mod$ corresponding to the ideal $R_{n>0}$ (cf. 1.6.1), i.e. as a *Zariski open immersion of* R_0-mod*-'spaces'*. However, unlike the commutative case, $\mathbf{Cone}_\Gamma(R)$ is not, in general, a subscheme of the affine scheme $R-mod \longrightarrow R_0-mod$.

6.8.1. Lemma. *Let $\mathcal{S} = \{S_i | i \in J\}$ be a family of left homogeneous Ore subsets of the algebra R. And let, for each $i \in J$, $\mathbb{S}_i$ be the Serre subcategory of $R-mod$ generated* *by all modules M such that any element of M is annihilated by some element of S_i. And let $\mathcal{S}_i$ be the preimage of $\mathbb{S}_i$ in $\mathfrak{gr}_\Gamma R - mod$.*

The following conditions are equivalent:

(a) The Serre subcategories $\{\mathbb{S}_i | i \in J\}$ provide a cover of the 'quasi-affine space' $\mathbf{Cone}_\Gamma(R)$; i.e. $\bigcap_{i\in J} \mathbb{S}_i = \mathcal{T}_+$.

(b) The family of Ore sets $\mathcal{S} = \{S_i | i \in J\}$ has the properties:

(i) For any $\gamma \in \Gamma$ and any $i \in J$, $S_i \cap R_{>\gamma} \neq \emptyset$.

(ii) If m is a left ideal of R such that $m \cap S_i \neq \emptyset$ for all $i \in J$, then $R_{>\gamma} \subseteq m$ for some $\gamma \in \Gamma$.

Proof. (a)⇔(b). The condition *(i)* is equivalent to the inclusion $\mathcal{T}_+ \subseteq \mathbb{S}_i$ for all $i \in J$. The condition *(ii)* says that $\mathcal{T}_+$ contains the intersection $\bigcap_{i\in J} \mathbb{S}_i$. Therefore $\mathcal{T}_+$ coincides with $\bigcap_{i\in J} \mathbb{S}_i$. ■

6.8.1.1. Remark. In [VW], a $\mathbb{Z}_+$-graded noetherian ring R such that there exists a family of *left and right* Ore sets $\mathcal{S} = \{S_i | i \in J\}$ satisfying the equivalent conditions of Lemma 6.8.1 is called *schematic*. Quite a few algebras of interest are schematic. We refer to [VW] for examples. ■

6.8.2. Proposition. *Any family $\mathcal{S} = \{S_i | i \in J\}$ of left homogeneous Ore subsets of R satisfying the conditions (i), (ii) of Lemma 6.8.1 determines an affine cover $\{\mathcal{S}_i | i \in J\}$ of $\mathbf{Cone}_\Gamma(R)$.*

Proof. The composition of π_* and the direct image of $u_i : \mathbf{Cone}_\Gamma(R)/\mathcal{S}_i \longrightarrow \mathbf{Cone}_\Gamma(R)$ equals to the composition of

$$\pi'_* \circ u_{i*} = (\pi' \circ u_i)_* : \mathbf{Cone}_\Gamma(R)/\mathcal{S}'_i \longrightarrow R - mod \qquad (1)$$

and the functor $\varphi_* : \mathfrak{gr}_\Gamma R - mod \longrightarrow R_0 - mod$ (cf. (1) in 6.7). Note that the category $\mathbf{Cone}_\Gamma(R)/\mathcal{S}'_i$ is naturally identified with the category $\mathfrak{gr}_\Gamma R - mod/\mathcal{S}_i$; so that the functor (1) becomes a right adjoint to the localization

$$Q_i : R - mod \longrightarrow R - mod/\mathcal{S}_i. \qquad (2)$$

Since (2) is a localization at a left Ore set S_i, the quotient category $R - mod/\mathcal{S}_i$ is equivalent to the category $S_i^{-1}R - mod$ of $S_i^{-1}R$-modules. Thus $R - mod/\mathcal{S}_i$ can be replaced by $S_i^{-1}R - mod$. And the localization Q_i can be identified with the tensoring $S_i^{-1}R\otimes_R$. Therefore a right adjoint functor to Q_i is exact. Since the functor $\varphi_* : R - mod \longrightarrow R_0 - mod$ is exact, we obtain the exactness of $\pi_* \circ u_{i*}$. ■

6.8.3. Lemma. *Let*

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ & u \searrow \quad \swarrow f & \\ & C & \end{array}$$

be a commutative diagram of continuous morphisms.

(a) If $u_* : U \longrightarrow C$ is exact (resp. u is almost affine) and the direct image functor f_* of f is faithful, then $g_* : U \longrightarrow X$ is exact (resp. g is almost affine).

(b) Suppose f^* is a localization. If u_* has a right adjoint, $u^!$ (resp. u is affine), then g_* has a right adjoint (resp. g is affine).

Proof. (a) Since $u_* \simeq f_* g_*$ and f_* is faithful, the compatibility of u_* with limits of a certain type of diagram implies the same property of g_*. Clearly the faithfulness of u_* implies that of g_*, hence if u is almost affine (i.e. u_* is exact and faithful), then g is almost affine.

(b) The condition 'f^* is a localization' means exactly that the direct image functor, f_* is fully faithful. Suppose the direct image functor u_* of u has a right adjoint, $u^!$. We claim that $u^! f_*$ is a right adjoint of the functor g_*. In fact, we have functorial isomorphisms: $U(-, u^! f_*(M)) \simeq C(u_*(-), f_*(M)) \simeq C(f_* g_*(-), f_*(M))$. Since the functor f_* is fully faithful, $C(f_* g_*(-), f_*(M)) \simeq C(g_*(-), M)$, hence the assertion ∎

6.8.3.1. Corollary. *Under conditions of 6.8.2, the morphisms $S_i^{-1}R - mod \longrightarrow Cone_\Gamma(R)$ are affine.*

6.8.4. Affine covers of Proj. Under the conditions of Proposition 6.8.2, the family of Ore sets $\mathcal{S} = \{S_i | i \in J\}$ determines an affine cover of $Proj_\Gamma(R) := \mathfrak{gr}_\Gamma R - mod/\mathcal{T}_+ \longrightarrow \mathfrak{gr}_\Gamma R - mod$. It follows from 6.8.3 that the morphisms $\mathfrak{gr}_\Gamma S_i^{-1} R - mod \longrightarrow Proj_\Gamma(R)$ are affine (cf. 6.8.3.1).

6.8.5. Example: noncommutative skew projective spaces. Let A be an arbitrary associative k-algebra. And let $\mathbf{q}$ denote a matrix $[q_{ij}]_{i,j \in J}$ with entrees in k such that $q_{ij} q_{ji} = 1$ for all $i, j \in J$. In particular, $q_{ii} = 1$ for all $i \in J$. To this data there corresponds a skew (or $\mathbf{q}-$)polynomial algebra $A_\mathbf{q}[\mathbf{x}]$, where $\mathbf{x}$ denotes the set of indeterminates $\{x_i | i \in J\}$. The defining relations are:

$$x_i x_j = q_{ij} x_j x_i \quad \text{for all } i, j \in J, \tag{1}$$

$$x_i r = r x_i \quad \text{for all } i \in J \text{ and } r \in R \tag{2}$$

Let $J = \{0, 1, \dots, r\}$. Set $\Gamma := \mathbb{Z}^{r+1}$; and let γ_i, $i = 0, 1, \dots, r$, denote the canonical generators of Γ. We provide Γ with a standard lexicographic preorder. Assigning to each x_i the parity γ_i, we turn the skew polynomial algebra $R := A_\mathbf{q}[\mathbf{x}]$ into a Γ-graded algebra with R_0=A. For any $i \in J$, set $S_i := \{x_i^n | n \geq 1\}$. The family $\mathcal{S} = \{S_i | i \in J\}$ satisfies the conditions (i), (ii) of Lemma 6.8.1. Therefore $\mathcal{S}$ determines, by Proposition 6.8, affine covers of the spaces $\mathbf{Proj}_\Gamma(R)$ and $\mathbf{Cone}_\Gamma(R)$. These covers have the usual 'classical' properties:

(a) The category $Cone_\Gamma(R)/\mathbb{S}'_i$ is equivalent to the category $A_\mathbf{q}[\mathbf{x}, x_i^{-1}] - mod$.

(b) Let Γ_i denote the quotient group $\Gamma/\mathbb{Z}\gamma_i \simeq \mathbb{Z}^r$. We have:

$$Proj_\Gamma(R)/\mathcal{S}'_i = \mathfrak{gr}_\Gamma R - mod/\mathcal{S}_i \simeq \mathfrak{gr}_\Gamma R[x_i, x_i^{-1}] - mod. \tag{3}$$

The category $\mathfrak{gr}_\Gamma R[x_i, x_i^{-1}] - mod$ in (3) is naturally equivalent to the category $\mathfrak{gr}_{\Gamma_i} A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$ of left Γ_i-graded modules over the skew polynomial algebra $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$. Here $\mathbf{x}/x_i$ denotes $\{x_j/x_i | j \in J, j \neq i\}$, and $\mathbf{q}_i$ denotes the matrix $[q_{ni} q_{nm} q_{ni}^{-1}]_{n,m \in J - \{i\}}$ (cf. [R], Example I.7.2.2.4).

Note that $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$ is the Γ_i-component of the algebra $A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]$ of the 'functions on $\mathbf{Cone}_\Gamma(R)/\mathbb{S}_i''$.

(c) One can see that the category $Proj_\Gamma(R)/\mathcal{S}_i'$ is naturally identified with the category $\mathfrak{gr}_{\Gamma_i} A_{\mathbf{q}_i}[\mathbf{x}/x_i]-mod$ and $Cone_\Gamma(R)/\mathbb{S}_i'$ with $\mathfrak{gr}_{\Gamma_i} A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]-mod$. And the canonical functor $Proj_\Gamma(R)/\mathcal{S}_i' \longrightarrow Cone_\Gamma(R)/\mathbb{S}_i'$ is induced by the tensoring by the algebra $A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]$ over its Γ_i-component $A_{\mathbf{q}_i}[\mathbf{x}/x_i] = A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]_0$.

Note however, that the natural morphisms $u_i : \mathfrak{gr}_{\Gamma_i} A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod \longrightarrow Proj_\Gamma(R)$ do not form an affine cover of $\mathbf{Proj}_\Gamma(R)$ if $r \geq 1$: the composition of u_{i*} with the direct image of the projection $\pi : Proj_\Gamma(R) \longrightarrow A-mod$ is not faithful, hence $\pi \circ u_i$ is not affine.

6.8.5.1. The projective q-'space' $\mathbf{P}^r_{\mathbf{q}}$. Let again $R = A_{\mathbf{q}}[\mathbf{x}]$, $\mathbf{x} = (x_0, x_1, \ldots, x_r)$. But take $\Gamma = \mathbb{Z}$ with the natural order; and set the parity of each x_i equal to 1. One can repeat with $\mathbf{Cone}_{\mathbb{Z}}(R)$ and $\mathbf{P}^r_{\mathbf{q}} := \mathbf{Proj}_{\mathbb{Z}}(R)$ the same pattern as with $\mathbf{Cone}_\Gamma(R)$ and $\mathbf{P}^r_\Gamma := \mathbf{Proj}_\Gamma(R)$. Only this time the quotient groups Γ_i will be trivial, and we obtain a picture very similar to the classical one: $\mathbf{P}^r_{\mathbf{q}}$ covered by $r+1$ affine spaces $A_{\mathbf{q}_i}[\mathbf{x}/x_i]-mod$, $i = 0, 1, \ldots, r$. The details are left to the reader.

Note that the categories P^r and $P^r_\Gamma := Proj_\Gamma(R)$ are not equivalent if $r \geq 1$.

6.9. Flag varieties of quantized enveloping algebras. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{P}$ denote the group of integral weights of $\mathfrak{g}$ and $\mathfrak{P}_+$ the semigroup of nonnegative integral weights. Let $\mathcal{R} = \oplus_{\lambda \in \mathfrak{P}_+} \mathcal{R}_\lambda$, where $\mathcal{R}_\lambda$ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathfrak{P}$ -graded algebra with the multiplication determined by the projections $\mathcal{R}_\lambda \otimes \mathcal{R}_\nu \longrightarrow \mathcal{R}_{\lambda+\nu}$, for all $\lambda, \nu \in \mathfrak{P}_+$. It is well known that the algebra $\mathcal{R}$ is isomorphic to the algebra of regular functions on the base affine space of $\mathfrak{g}$. Recall that $Y = G/U$, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The $Cone_\Gamma(\mathcal{R})$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $Proj_\Gamma(\mathcal{R})$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

Let now $\mathfrak{g}$ be a semisimple Lie algebra over a field k of zero characteristic and $U_q(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathfrak{P}$ -graded algebra $\mathcal{R} = \oplus_{\lambda \in \mathfrak{P}_+} \mathcal{R}_\lambda$ the same way as above. This time, however, the algebra $\mathcal{R}$ is not commutative. Following the classical example (and identifying spaces with categories of quasi-coherent sheaves on them), we call $\mathbf{Cone}_\Gamma(\mathcal{R})$ *the quantum base affine space* and $\mathbf{Proj}(\mathcal{R})$ the *quantum flag variety* of $\mathfrak{g}$.

6.9.1. An affine cover of the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathfrak{P}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ formed by the vectors of the weight $w\lambda$. Set $S_w := \{k^* e^\lambda_{w\lambda} | \lambda \in \mathfrak{P}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in $\mathcal{R}$. The Ore sets $\{S_w | w \in W\}$ determine a locally affine cover of the quantum base affine space $\mathbf{Cone}_\Gamma(\mathcal{R})$ and the quantum flag variety $\mathbf{Proj}_\Gamma(\mathcal{R})$ of $\mathfrak{g}$. This cover enjoys properties similar to the properties (a)–(c) of the canonical cover of a 'projective space' and its cone (cf. 6.8.5). Namely, $Cone_\Gamma(\mathcal{R})/\mathbb{S}_w$ is naturally equivalent to $S_w^{-1}\mathcal{R} - mod$ and $Proj_\Gamma(\mathcal{R})/\mathcal{S}_w$ is naturally equivalent to $(S_w^{-1}\mathcal{R})_0 - mod.$

7. Abelianization.

Fix a base $\mathfrak{C} = (\mathcal{A}^\sim, C, \Phi)$. And let $\beta = (\beta_{L,M} : L \otimes M \longrightarrow M \otimes L | L, M \in Ob\mathcal{A})$ be a symmetry of the monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \otimes, \mathbf{1})$.

7.1. Abelianization of affine schemes. Denote by $CAlg_\beta\mathcal{A}^\sim$ the full subcategory of the category $Alg\mathcal{A}^\sim$ generated by β-commutative algebras.

7.1.1. Lemma. *The inclusion functor $CAlg_\beta\mathcal{A}^\sim \longrightarrow Alg\mathcal{A}^\sim$ has a left adjoint, $\mathfrak{a}_\beta$.*

Proof. The functor $\mathfrak{a}_\beta$ assigns to any algebra $R^\sim = (R, m)$ the cokernel R_β of the pair $m, m \circ \beta_{R,R} : R \otimes R \longrightarrow R$ with the unique multiplication $m_\beta : R_\beta \otimes R_\beta \longrightarrow R_\beta$ such that the canonical epimorphism $R \longrightarrow R_\beta$ is an algebra morphism. ■

We call $\mathfrak{a}_\beta$ the *β-abelianization functor.*

Denote by $\mathbf{Aff}^\beta_{\mathcal{A}^\sim}$ the full subcategory of the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes generated by *β-commutative affine schemes*, i.e. affine schemes isomorphic to $\mathbf{Spec}(R^\sim)$ for a β-commutative algebra $R^\sim$.

7.1.2. Proposition. *(a) The inclusion functor $\mathbf{Aff}^\beta_{\mathcal{A}^\sim} \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$ has a right adjoint.*

(b) The functor $\mathfrak{Ab}_\beta$ preserves arbitrary coproducts and limits.

(c) For any pair $S^\sim \longleftarrow R^\sim \longrightarrow T^\sim$ of β-commutative algebra morphisms, the fiber product of the corresponding β-commutative affine schemes is isomorphic to the affine scheme $\mathbf{Spec}(S^\sim \otimes^\beta_{R^\sim} T^\sim)$.

(d) For any affine $\mathcal{A}^\sim$-scheme X, the adjunction morphism $J_\beta \circ \mathfrak{Ab}_\beta(X) \longrightarrow X$ is a closed immersion.

Proof. (a) The assertion follows from 7.1.1.

(b) The functor $\mathfrak{Ab}_\beta$ preserves limits as any functor having a left adjoint.

The product of any family of β-commutative algebras (taken in the category $Alg\mathcal{A}^\sim$ of all algebras in $\mathcal{A}^\sim$) is a β-commutative algebra, i.e. the β-abelianization functor $\mathfrak{a}_\beta$: $Alg\mathcal{A}^\sim \longrightarrow CAlg\mathcal{A}^\sim$ is compatible with arbitrary products. By duality, $\mathfrak{Ab}_\beta$ preserves arbitrary coproducts.

(c) The pull-forward of a pair $S^\sim \longleftarrow R^\sim \longrightarrow T^\sim$ of β-commutative algebra morphisms is isomorphic to $\mathfrak{a}_\beta(S^\sim \star_{R^\sim} T^\sim)$, and the latter is isomorphic to $S^\sim \otimes_{R^\sim} T^\sim$. Hence the assertion.

(d) This follows from the fact that the canonical morphism from an algebra $R^\sim = (R, \mu)$ to its β-abelianization, $R^\sim_\beta$, is a cokernel of the pair of morphisms $\mu, \mu \circ \beta_{R,R}$: $R \otimes R \longrightarrow R$ (cf. 5.5.1). ■

7.2. β-commutative schemes. The category $\mathbf{Sch}^\beta_{\mathcal{A}^\sim}$ of β-commutative $\mathcal{A}^\sim$-schemes is defined the same way as the category of $\mathcal{A}^\sim$-schemes with the only difference: the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes is replaced by its subcategory $\mathbf{Aff}^\beta_{\mathcal{A}^\sim}$ of β-commutative schemes.

7.3. The β-abelianization of schemes. Let F be a functor $\mathbf{Aff}^{op}_{\mathcal{A}^\sim} \longrightarrow \mathbf{Sets}$. The *$\beta$-abelianization of the functor F, F^β*, is the composition of F with (the opposite to) the inclusion functor $\mathbf{Aff}^\beta_{\mathcal{A}^\sim} \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$.

Let X be a scheme. Suppose that the β-abelianization of the functor $h_X : \mathbf{Sch}^{op}_{\mathcal{A}^\sim} \longrightarrow \mathbf{Sets}$, $Y \longmapsto Sch_{\mathcal{A}^\sim}(Y, X)$, is representable by a β-commutative scheme X_β. The scheme X_β will be called a β-abelianization of X.

By 7.1.2, the β-abelianization of a functor represented by an affine scheme $\mathbf{X}$ is represented by the abelianization of $\mathbf{X}$.

7.3.1. Proposition. *Let X_β be a β-abelianization of a $\mathcal{A}^\sim$-scheme X. The canonical morphism $X_\beta \longrightarrow X$ is a closed immersion.*

Proof. The assertion follows from 7.1.2. ■

7.4. Examples. The following examples show that all noncommutative varieties discussed in Section 6 have β-abelianizations.

7.4.1. β-abelianization of vector fibers. Let $E \in Ob\mathcal{A}$ and be as in 6.1.1. Then the tensor algebra $T(E)$, hence the vector fiber $\mathbf{V}(E) = \mathbf{Spec}(T(E))$, are defined. The β-abelianization $\mathbf{V}_\beta(E)$ of $\mathbf{V}(E)$ is $\mathbf{Spec}(S_\beta(E))$, where $S_\beta(E)$ is the β-symmetric algebra of E. It follows that the β-abelianization of an admissible pair (L, M) of objects of the category C is represented by the β-commutative affine scheme $\mathbf{V}_\beta(L^\wedge M) := \mathbf{Spec}(S_\beta(L^\wedge M))$.

7.4.1.1. Affine q-'spaces'. Let k be a commutative ring, and let $\mathbf{q}$ be a matrix $(q_{ij})_{0\le i,j\le r}$ with entrees in k such that $q_{ij}q_{ji} = 1$ for all $1 \le i, j \le r$. The matrix $\mathbf{q}$ defines a symmetry, $\beta_\mathbf{q}$, of the monoidal category $\mathcal{A}^\sim = \mathfrak{gr}_{\mathbb{Z}^{r+1}}k - mod^\sim$ of $\mathbb{Z}^{r+1}$-graded k-modules. Let γ_i denote the i-th standard generator of the group $\mathbb{Z}^{r+1}$, and let P_i be of a free k module of rank 1 generated by an element of degree γ_i. Set $E = \oplus_{0\le i,j\le r} P_i$. The $\beta_\mathbf{q}$-abelianization of $\mathbf{V}(E)$ is the $\mathbf{q}$-'space' $\mathbf{Spec}(k_\mathbf{q}[\mathbf{x}])$ (cf. 6.8.5).

7.4.1.2. Projective q-'spaces'. Let $E \in Ob\mathfrak{gr}_{\mathbb{Z}^{r+1}}k - mod$ be the same as in 7.4.1.1. The $\beta_\mathbf{q}$-abelianization of the projective fiber $\mathbf{P}(E) := \mathbf{Proj}(T(E))$ is isomorphic to the $\mathbf{q}$-projective space $\mathbf{P}^r_\mathbf{q} := \mathbf{Proj}(k_\mathbf{q}[\mathbf{x}])$ of 6.8.5.1.

7.4.2. The β-commutative group scheme $GL_\beta(V)$.

7.4.2.1. Proposition. *Let V be an object of C such that the pair (V, V) is admissible. The functor*

$$Aut^\beta_V : (\mathbf{Aff}^\beta_{\mathcal{A}^\sim})^{op} \longrightarrow \mathbf{Sets}, \quad (\mathbf{X}) \longmapsto Aut(f^*_\mathbf{X}(V))$$

is representable by an affine β-commutative $\mathcal{A}^\sim$-scheme in groups.

Proof. The assertion follows from 6.4.1 and 7.3.1. ■

7.4.3. β-Grassmannians. Let L, M be objects of the category C.

7.4.3.1. Proposition. *Under conditions of 6.5.2, the β-abelianization of the functor $Grass_{L,M}$ is representable by a β-commutative scheme $Grass^\beta_{L,M}$.*

Proof. The assertion can be proved by following step by step the argument of 6.5.3. The details are left to the reader. ■

The β-commutative scheme $Grass^{\beta}_{L,M}$ will be called the *L-th β-Grassmannian of M.*

7.4.3.2. The usual Grassmannians. Let $\mathcal{A}^{\sim}$ be the monoidal category of quasi-coherent sheaves on a scheme $\mathbf{S} = (S, \mathcal{O})$, β the standard symmetry. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{A}^{\sim})$. Fix a quasi-coherent sheaf M, and take L equal to the direct sum of n copies of $\mathcal{O}_S$. Then $\mathbf{Grass}^{\beta}_{M,L}$ is the usual n-the Grassmannian of the quasi-coherent sheaf M in the sense of [GrD], 9.7: $Grass^{\beta}_{M,L} = Grass_n(M)$.

7.4.4. β-flag varieties. Let $M \in ObC$, and let $\mathbf{L} = (L_i)_{1 \le i \le p}$ be an increasing sequence of objects of C such that the pairs (M, L_i) and (L_i, L_i) are admissible for all i.

7.4.4.1. Proposition. *Under conditions of 6.6.3, the β-abelianization of the functor $Fl_{\mathbf{L},M}$ is representable by a β-commutative scheme $Fl^{\beta}_{\mathbf{L},M}$.*

Proof. The assertion follows from the corresponding fact about Grassmannians (cf. 7.4.3.1). ■

7.4.4.2. The usual flag varieties. Let $\mathcal{A}^{\sim}$ be the monoidal category of quasi-coherent sheaves on a scheme $\mathbf{S} = (S, \mathcal{O})$ with the standard symmetry β. And let $\mathfrak{C} = (\mathcal{A}, \mathcal{A}^{\sim})$. Fix a quasi-coherent sheaf M on $\mathbf{S}$. Fix an increasing sequence $\mathbf{m} = (m_i)_{1 \le i \le p}$ of nonnegative integers. Set $\mathbf{L} = (L_i)_{1 \le i \le p}$, where $L_i = \mathcal{O}^{m_i}_S$ – the direct sum of m_i copies of the structure sheaf $\mathcal{O}_S$. Then $\mathbf{Fl}^{\beta}_{\mathbf{L},M}$ is the usual flag variety of the type $\mathbf{m}$ in the quasi-coherent sheaf M in the sense of [GrD], 9.9.

7.5. β-nilpotent schemes. For simplicity, we assume that all categories in this subsection are abelian. To any pair of closed subschemes Y and Z of a $\mathfrak{C}$-scheme $(\mathbf{X})$, one can assign their *Gabriel product*, $Y \bullet Z$, defined as follows. We identify $Qcoh_Y$ and $Qcoh_Z$ with full subcategories of $Qcoh_X$. Then $Qcoh_{Y \bullet Z}$ is the full subcategory of X generated by all $M \in ObX$ such that there exists an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ with $M' \in ObY$ and $M'' \in ObZ$. The subcategory $Qcoh_{Y \bullet Z}$ defines a closed subscheme ([R], III.6.2.1). In particular, for any closed subscheme Y of a $\mathfrak{C}$-scheme X has *thickennings* $Y^{(n)}$, $n \ge 1$ defined by $Y^{(1)} = Y$, and $Y^{(n+1)} = Y^{(n)} \bullet Y$.

An $\mathcal{A}^{\sim}$-scheme X will be called a *β-nilpotent $\mathcal{A}^{\sim}$-scheme of degree* $\le n$ if there exists a closed β-commutative subscheme Y of X such that X coincides with the n-th thickenning of Y. We denote by $\mathbf{Sch}^{\beta,n}_{\mathcal{A}^{\sim}}$ the full subcategory of the category $\mathbf{Sch}_{\mathcal{A}^{\sim}}$ generated by β-nilpotent schemes of degree $\le n$. Clearly $\mathbf{Sch}^{\beta}_{\mathcal{A}^{\sim}} = \mathbf{Sch}^{\beta,1}_{\mathcal{A}^{\sim}}$ and $\mathbf{Sch}^{\beta,n}_{\mathcal{A}^{\sim}} \subseteq \mathbf{Sch}^{\beta,m}_{\mathcal{A}^{\sim}}$ if $n \le m$. We denote by $\mathbf{Aff}^{\beta,n}_{\mathcal{A}^{\sim}}$ the category of affine β-nilpotent schemes of degree $\le n$: $\mathbf{Aff}^{\beta,n}_{\mathcal{A}^{\sim}} = \mathbf{Aff}_{\mathcal{A}^{\sim}} \cap \mathbf{Sch}^{\beta,n}_{\mathcal{A}^{\sim}}$.

For any functor $F : \mathbf{Sch}^{op}_{\mathcal{A}^{\sim}} \longrightarrow \mathbf{Sets}$, denote by $F^{\beta,n}$ the composition of F with the (opposite to the) inclusion functor $\mathbf{Sch}^{\beta,n}_{\mathcal{A}^{\sim}} \longrightarrow \mathbf{Sch}^{\sim}_{\mathcal{A}}$. We call the functor $F^{\beta,n}$ the *β-nilpotezation of the functor F of degree n*, or simply *β-nilpotezation of F* if $n = \infty$.

Let X be a $\mathcal{A}^{\sim}$-scheme. Suppose that the β-nilpotezation of degree n of the functor $h_X : \mathbf{Sch}^{op}_{\mathcal{A}^{\sim}} \longrightarrow \mathbf{Sets}$, $Y \longmapsto \mathbf{Sch}_{\mathcal{A}^{\sim}}(Y, X)$, is representable by a $\mathcal{A}^{\sim}$-scheme $X_{\beta,n}$. The scheme $X_{\beta,n}$ will be called a *β-nilpotezation of X of degree n.*

7.5.1. Proposition. *(a) If a β-abelianization of a $\mathcal{A}^{\sim}$-scheme X exists, then there exist β-nilpotezations of X all degrees.*

(b) The canonical morphism $X_{\beta,n} \longrightarrow X$ is a closed immersion.

(c) For all n, the functor $\mathfrak{N}_{\beta,n}$ of β-nilpotezation of degree n, $\mathfrak{N}_{\beta,n} : X \longmapsto X_{\beta,n}$ preserves arbitrary coproducts and limits taken in $\mathbf{Sch}_{\mathcal{A}^\sim}$.

Proof. (a) & (b) Let X_β be a β-abelianization of a $\mathcal{A}^\sim$-scheme X, and let $X_\beta \longrightarrow X$ be the canonical immersion. The n-th neighborhood of X_β in X is exactly the β-nilpotezation of X of degree n, $X_\beta^{(n)} = X_{\beta,n}$.

Since $X_\beta \longrightarrow X$ is a closed immersion, $X_{\beta,n} = X_\beta^{(n)} \longrightarrow X$ is a closed immersion. The latter implies that $X_{\beta,n}$ is a subscheme of the $\mathcal{A}^\sim$-scheme X.

(c) The compatibility with limits follows from general properties of representable functors. The compatibility the functor $\mathfrak{N}_{\beta,n}$ with coproducts is follows from the description of coproducts. ■

7.5.2. Proposition. *(a) For any affine $\mathcal{A}^\sim$-scheme X, there exist β-nilpotezation of X of degree n for all n.*

(b) All β-nilpotezations of an affine $\mathcal{A}^\sim$-scheme are affine. In particular, the inclusion functor $\mathbf{Aff}_{\mathcal{A}^\sim}^{\beta,n} \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}$ has a right adjoint, $\mathfrak{N}_{\beta,n} : \mathbf{Aff}_{\mathcal{A}^\sim} \longrightarrow \mathbf{Aff}_{\mathcal{A}^\sim}^{\beta,n}$.

Proof. (a) The assertion (a) follows from 7.1.2, 7.3.1, and 7.5.1.

(b) Let X be an affine $\mathcal{A}^\sim$-scheme. By 7.5.1, the canonical morphism $X_{\beta,n} \longrightarrow X$ is a closed immersion. The assertion now follows from the fact that any closed subscheme of an affine $\mathcal{A}^\sim$-scheme is affine. ■

7.5.3. Examples. Any of noncommutative schemes discussed in Section 6 has β-nilpotezations of all degrees. This follows (thanks to 7.5.1(a)) from the fact that all these schemes have β-abelianizations.

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