Chapter II

The Left Spectrum and Irreducible Representations of 'Small' Quantized and Classical Rings.

Introduction

The first Heisenberg and Weyl algebras and $U(sl(2))$ - the universal enveloping algebra of the Lie algebra $sl(2)$ - are the most important "small" algebras of the preceeding to quantum group epoch of mathematical physics and representation theory. Quantum group activity has already produced a lot more. The following list contains only principal examples of "small" quantum algebras:

(a) Quantum plane (or, better, *q-plane*) $k_q[x.y]$ is an associative algebra over a field k generated by x and y satisfying the relation:

$$xy = qyx, \quad q \in k^*. \tag{1}$$

(b) The algebra of q-differential operators $\mathbb{D}_{q,h} = \mathbb{D}_{q,h}[x, y]$ which is defined by the relation:

$$xy - qyx = h. \tag{2}$$

(c) Quantum Heisenberg algebra $\mathbb{H}_q$ generated over the field k by x, y, z subject to the following relations:

$$xz = qzx, \quad zy = qyz; \quad xy - qyx = z. \tag{3}$$

(d) The first quantum Weyl algebra $\mathbb{W}_{q,1}$ which is obtained from $\mathbb{H}_q$ by adding the relations:

$$(xy - q^{-1}yx)z = 1 = z(xy - q^{-1}yx). \tag{4}$$

(e) The quantum enveloping algebra of the Lie algebra $sl(2)$, $U_q(sl(2))$, defined by the relations:

$$xz = qzx, \quad zy = qyz; \quad xy - yx = \frac{z - z^{-1}}{q - q^{-1}} \tag{5}$$

(f) The coordinate ring of quantum 2×2 *matrices* $M_q(2)$ which has generators x, y, u, v satisfying the relations

$$xu = qux, \quad xv = qvx, \quad qyu = uy, \quad qyv = vy, \quad uv = vu,$$

$$xy - yx = (q - q^{-1})uv. \tag{6}$$

(g) The coordinate algebra $A(SL_q(2))$ otherwise called the algebra of functions of the quantum group $SL(2)$, is generated by x, y, u, v subject to the relations:

$$xu = qux, \quad xv = qvx, \quad qyu = uy, \quad qyv = vy, \quad uv = vu,$$

$$xy - quv = 1 = yx - q^{-1}uv. \tag{7}$$

(h) Twisted $SL(2)$ *group,* $\mathbb{W}_\nu(sl(2))$, by Woronowicz [W] which is defined by relations:

$$xz - \nu^4 zx = (1+\nu^2)x, \quad zy - \nu^4 yz = (1+\nu^2)y,$$

$$xy - \nu^2 yx = \nu z. \tag{8}$$

The problems of determining the irreducible representations of the Weyl algebra and of the Lie algebra $sl(2)$ were for a long time regarded as hopeless, and their solution by R. Block [B1], [B2] is still remembered as a real 'break through' which it, certainly, was.

One of the goals of this chapter is to obtain the representation theory of all listed above algebras. The way we approach to the problem is based on the developed in Chapter I noncommutative spectral theory and on the following observation:

all the algebras above, and a number of others, belong to the class of *hyperbolic rings* (which was first introduced in [R3]).

Given an automorphism θ of a commutative ring A, and an element ξ of A, the *hyperbolic ring* $A\{\theta, \xi\}$ is defined as the ring generated by A and the two indeterminates x, y satisfying the relations:

$$xa = \theta(a)x, \quad ay = y\theta(a) \quad \text{for all } a \in A, \tag{9}$$

$$xy = \xi, \quad yx = \theta^{-1}(\xi). \tag{10}$$

('hyperbolic' is due to the relation (10)). As the reader shall see, the hyperbolic rings turn out to be convenient enough to allow a complete description of their left spectrum.

The left spectrum is a natural extension of the set of left maximal ideals. And, in many cases, it is not difficult to single out left maximal ideals ("closed points") from the description of the left spectrum. For instance, we recover the classification by R. Block of irreducible representations of the first Weyl algebra [B1], [B2] using just general facts about relations between the Krull dimension and the hight of points of the left spectrum established in Chapter VI.

Note that R. Block studied irreducible representations of $U(sl(2))$ and of the enveloping algebra of the two-dimensional nonabelian algebra Lie [B2] by using the homomorphisms of these algebras to the first Weyl algebra A_1 and the already obtained classification of the irreducible representations of A_1. Here we first get the classification of the left spectrum of general skew polynomial and hyperbolic rings, and then apply it to special cases. As a result, the classification we get is given in terms of natural for each of the rings in question parameters.

Section 0 provides, for readers' convenience, preliminaries on the spectrum of abelian categories (which is a topic of Chapter III).

In Section 1, we study the left spectrum of the ring of skew polynomials over a commutative ring. The specialization of general facts gives a complete description of the left spectrum of the universal enveloping algebra of the 2-dimensional noncommutative Lie algebra over a field of characteristic zero and the quantum plane, $k_q[x, y]$, when q is not a root of one.

To cover the root of unity and positive characteristic cases, we introduce, in Section 2, *restricted skew polynomial* rings and study their left spectrum. A restricted skew polynomial ring is given by the relations

$$xa = \theta(a)x \text{ for all } a \in A, \text{ and } x^n = u,$$

where θ is an automorphism of A such that $\theta^n = id$, x is an indeterminate, and u a fixed element of A.

Section 3 is the heart of the chapter. It contains an almost complete description of the left spectrum of hyperbolic rings and restricted hyperbolic rings (the latter are defined when $\theta^n = id$ for some n). The complete description is out of reach of the technique used here. We shall get it in Chapter IV, and even in a much more general setting which gives an access to some important classes of "non-small" algebras, like the Weyl and Heisenberg algebras of arbitrary ranks and their (quantum) deformations.

The results of Section 3 allow to describe the left spectrum of all listed above hyperbolic rings (cf. *(c) - (h)*) and of a number of others. We sketch their spectral pictures in Sections 4 and in 'Complementary facts and examples' which a reader can regard as a kind of a handbook on representation theory of important (not only for mathematical physics) examples of hyperbolic rings of small GK-dimension.

A pure luck is that most of 'small' rings of interest are hyperbolic.

0. Preliminaries: the spectrum of an abelian category.

The proofs of the assertions of this section can be found in Chapter III.

We shall need a definition of the left spectrum in categorical terms.

Let $\mathcal{A}$ be an abelian category (in this chapter, $\mathcal{A}$ is the category $R-mod$ of left R-modules); and let M, N be objects of $\mathcal{A}$. We shall write $M \succ N$ if there exists a diagram

$$(l)M \longleftarrow L \longrightarrow N,$$

where $(l)M$ is the direct sum of l copies of M; the first arrow is a monomorphism and the second arrow is an epimorphism. In other words, $M \succ N$ iff N is a subquotient of the direct sum of a finite number of copies of M.

Denote by $Spec\mathcal{A}$ the collection of all the objects M of $\mathcal{A}$ such that $N \succ M$ for any nonzero subobject N of M.

0.1. Lemma. *The relation $\succ$ is a preorder in $Ob\mathcal{A}$. In particular, $\succ$ determines an equivalence relation, $\approx$, in $Spec\mathcal{A}$.*

Proof. See Lemma 1.1.1 in Chapter III. ■

Denote the (ordered) set of equivalence classes $Spec\mathcal{A}/\approx$ by $\mathbf{Spec}\mathcal{A}$.

0.2. Remarks. a) It follows from the definition that $Spec\mathcal{A}$ contains all simple objects of the category $\mathcal{A}$.

b) An equivalence of abelian categories, $\mathcal{A} \longrightarrow \mathcal{B}$, induces a bijection of $Spec\mathcal{A}/\approx$ onto $Spec\mathcal{B}/\approx$. ■

0.3. Proposition. *Let $\mathcal{A}$ is the category $R-mod$ of left modules over a ring R. Then the map $Spec_l R \longrightarrow Ob\mathcal{A}$ assigning to a left ideal p the quotient module R/p induces a bijection of the sets of equivalence classes*

$$\{\langle p\rangle \mid p \in Spec_l R\} := \mathbf{Spec}_l R \longrightarrow \mathbf{Spec}\mathcal{A}.$$

Proof. The assertion follows from Proposition 4.2 in Chapter III. ■

0.4. Corollary. *Let rings R and R' be Morita equivalent; i.e. there is an equivalence between the categories of left modules, R-mod and R'-mod. Then there is a bijection of* $\mathbf{Spec}_l R$ *onto* $\mathbf{Spec}_l R'$.

0.5. Corollary. *The set $Max_l R$ of left maximal ideals of R is contained in $Spec_l R$.*

This follows from Proposition 0.3 and Remark 0.2 a). ■

1. The left spectrum of the ring of skew polynomials. Quantum plane.

Let A be a commutative ring with unity, and let ϑ be an automorphism of A. The associative ring $A[x;\vartheta]$ of *ϑ-skew polynomials* is generated by the ring A and the indeterminate x subject to the relations:

$$xa = \vartheta(a)x \quad \text{for every } a \in A.$$

1.1. Examples. Let $A = k[y]$. A generic automorphism, ϑ, of the k-algebra A is defined by $\vartheta(y) = q(y+\alpha)$, where $q \in k^*$ and $\alpha \in k$. Consider two special cases:

(a) Let $\alpha = 0$. Then $A[x;\vartheta]$ is the k-algebra generated by x and y which satisfy the relation:

$$xy = qyx.$$

This algebra is called *quantum plane* and is, usually, denoted by $k_q[x,y]$.

(b) Let now $q = 1$. Then the ring $A[x;\vartheta]$ is generated by x, y satisfying the relation:

$$xy = yx + \alpha x.$$

Denote this algebra by $U_2(k,\alpha)$. Clearly $U_2(k,0) = k[x,y]$. If $\alpha \neq 0$, then the k-algebra $U_2(k,\alpha)$ is the enveloping algebra of the (unique up to isomorphism) two-dimesional non-abelian Lie algebra.

(c) The generic case, more explicitly, the case $q \neq 1$, is, again, a quantum plane. In fact, $\vartheta(y-\alpha/(1-q)) = q(y-\alpha/(1-q))$ which means that the change of variables $z \longmapsto (y-\alpha/(1-q))$ establishes a k-algebra isomorphism of the quantum plane $k_q[x,z]$ and the algebra $A[x;\vartheta]$. ■

1.2. The left spectrum of $A[x;\vartheta]$ and the prime spectrum of A. We begin with the following observation:

$A[x;\vartheta]x$ is a two-sided ideal, and the natural map $A \longrightarrow A[x;\vartheta]/A[x;\vartheta]x$ is an isomorphism.

Therefore $Spec_l A[x;\vartheta] = V_l(x) \bigcup U_l(x)$, where the closed subset $V_l(x) = \{\mathbf{p} \mid x \in \mathbf{p}\} = \{\mathbf{p} \mid A[x,\vartheta]x \subseteq \mathbf{p}\}$ is naturally homeomorphic to $SpecA$, and the open subset $U_l(x) = \{\mathbf{p} \mid x \notin \mathbf{p}\} = U_l(A[x,\vartheta]x)$ is going to be a subject of our investigation.

Note that $U_l(x)$ is homeomorphic to a subspace of $Spec_l A[x,x^{-1};\vartheta]$, where $A[x,x^{-1};\vartheta]$ is the module $A[x,x^{-1}]$ of Laurent polynomials with the multiplication (uniquely) determined by the requirement

$$xa = \vartheta(a)x \text{ for any } a \in A \quad (\text{hence } x^{-1}a = \vartheta^{-1}(a)x^{-1}).$$

Suppose now that the ring A is noetherian. Fix $\mathbf{p} \in Spec_l A[x,x^{-1};\vartheta]$; and set $p = \mathbf{p}\bigcap A$. Let $(p:a)$ be a maximal (with respect to the inclusion) element of the set $\Omega_{p'} := \{(p:b) \mid b \in A - p\}$. According to Lemma 0.5.1, the ideal $(p:a)$ is prime. Thus, replacing $\mathbf{p}$ by the equivalent to $\mathbf{p}$ ideal $(\mathbf{p}:a)$, we can assume that the ideal $p = \mathbf{p}\bigcap A$ of the ring A is prime.

In the non-noetherian case, we restrict our study to the subset of those ideals $\mathbf{p} \in Spec_l A[x,x^{-1};\vartheta]$ for which $\mathbf{p}\bigcap A$ is a prime ideal in A.

1.3. The left ideals of $A[x,\vartheta]$ over primes in A. We assume now that A is an arbitrary commutative ring, and shall study left ideals $\mathbf{p}$ in $A[x,\vartheta]$ such that the intersection $\mathbf{p}\bigcap A$ is a prime ideal in A.

It is convenient to distinguish the following alternatives:
(a) $p = \mathbf{p}\bigcap A = \{0\}$;
(b) *p is non-trivial and ϑ-stable;*
(c) *p is not invariant under ϑ^n for any n.*
Thus, the only remaining possibility is:
(d) *p is not ϑ-stable, but p is invariant under ϑ^n for some n.*
Consider each of these cases.

1.3.1. The stable cases. Which are the cases (a) and (b) above.
(a) Let $p = \mathbf{p}\bigcap A = \{0\}$; in particular, A is a domain.
Then we can take the localization Q_A of the ring $B := A[x,x^{-1};\vartheta]$ at the set $A - \{0\}$. Note that $A - \{0\}$ is an Ore set; hence $Q_A B$ is isomorphic to the ring $K(A)[x,x^{-1};\vartheta']$, where $K(A)$ is the field of fractions of the ring A, and ϑ' is the (unique) extension of the automorphism ϑ onto the field $K(A)$.

It is easy to check that $K(A)[x;\vartheta]$ is an euclidean domain (for any skew field $K(A)$). In particular, $K(A)[x,\vartheta]$ is a left and right principal ideal domain. Therefore (cf. Proposition 0.4.1) any ideal from $Spec_l K(A)[x;\vartheta']$ is equivalent to a left maximal ideal, and any left maximal ideal is of the form $K(A)[x,\vartheta']g$, where g is an irreducible element (polynomial) of $K(A)[x,\vartheta']$. Clearly
$Spec_l K(A)[x,x^{-1};\vartheta]$ is $Spec_l K(A)[x,;\vartheta]$ without one point – the (two- sided) maximal ideal $K(A)[x,\vartheta']x$.

(b) Suppose now that $p := \mathbf{p}\bigcap A$ is a nonzero ϑ-invariant prime ideal.
Then ϑ induces an automorphism, ϑ', of the quotient ring $A' = A/p$. The surjection $A \longrightarrow A'$ induces an epimorphism,

$$\varphi : A[x, \vartheta] \longrightarrow A'[x, \vartheta']$$

such that $\varphi(x) = x$. The image, $\mathbf{p}'$, of the ideal $\mathbf{p}$ belongs to the left spectrum of $A'[x, \vartheta']$; and $\mathbf{p}' \bigcap A' = \{0\}$.

Hence there exists an element $g = g(x) \in A[x, \vartheta]$ such that $g' = \varphi(g)$ is an irreducible element in $K(A')[x, \vartheta']$ and $\mathbf{p}$ is the preimage of the maximal ideal $\mathbf{p}' = K(A')[x, \vartheta']g'$ under the canonical ring morphism

$$A[x, \vartheta] \longrightarrow K(A')[x, \vartheta']$$

(cf. *(a)* above).

(c) Consider now the most interesting, case: *the ideal* $p = \mathbf{p} \bigcap A$ *is not invariant under the automorphism* ϑ.

1.3.2. Lemma. *Let* $\mathbf{p}$ *be a left ideal of the ring* $A[x, x^{-1}; \vartheta]$ *such that* $\mathbf{p} \bigcap A$ *is a prime ideal in* A. *Suppose that* $\mathbf{p}$ *contains a polynomial*

$$f(x) = \sum x^m g_m \in A[x; \vartheta],$$

some of the coefficients g_m *of which do not belong to* $\mathbf{p}$. *Then there exists an integer* ν *such that*

$$1 \leq \nu \leq n = deg(f), \quad and \quad \vartheta^{-\nu}(\mathbf{p} \cap A) \subseteq \mathbf{p} \cap A.$$

Proof. Denote the intersection $\mathbf{p} \bigcap A$ by p. Choose a polynimial $f(x) = \sum x^m g_m \in \mathbf{p}$ of minimal degree among the polynomials from $\mathbf{p}$ with some coefficients from $A - p$. We can (and will) assume from the very beginning that all the nonzero coefficients of the polynomial f do not belong to the ideal p.

Let λ be an arbitrary nonzero element of the ideal p. It is easy to see that

$$\lambda f(x) - f(x)\lambda = x(\sum x^{m-1} \vartheta^{-m}(\lambda) g_m - \sum x^{m-1} g_m \lambda)$$

Since $\lambda f(x) - f(x)\lambda$ and $\sum x^{m-1} g_m \lambda$ are elements of $\mathbf{p}$, the polynomial

$$\vartheta\hat{}(\lambda) f(x) := \sum x^{m-1} \vartheta^{-m}(\lambda) g_m$$

also belongs to $\mathbf{p}$. But $deg(\vartheta\hat{}(\lambda) f) < deg(f)$. Therefore, thanks to the minimality of $deg(f)$, all the coefficients, $\vartheta^{-m}(\lambda) g_m$, of the polynomial $\vartheta\hat{}(\lambda) f$ are elements of the ideal p. Since $p \in SpecA$, and, by hypothesis, the nonzero coefficients of the polynomial f belong to $A - p$, the ideal p is invariant under the automorphism ϑ^{-m} if the coefficient g_m is nonzero. ∎

1.3.3. Corollary. *Let* $\mathbf{p}$ *be a left ideal of the ring* $A[x, x^{-1}; \vartheta]$ *such that* $p = \mathbf{p} \bigcap A$ *is a prime ideal in* A. *Suppose that* p *is invariant under* ϑ^n *for some* $n \geq 2$, *but not invariant under* ϑ^m *for any* $1 \leq m < n$. *Then every polynomial in* $\mathbf{p}$ *of degree less than* n *belongs to* $\mathbf{p} \bigcap A[x, x^{-1}, \vartheta]p$.

1.3.4. Proposition. *Let* $\mathbf{p}$ *be a left ideal in* $A[x, \vartheta]$ *such that* $p := \mathbf{p} \bigcap A$ *is a nonzero prime ideal, which is not* ϑ^m*-stable for any integer* m*. Then*

1) If the ideal $\mathbf{p}$ *does not contain* x^n *for any* $n \geq 1$*, then* $\mathbf{p}$ *is generated by* p *:* $\mathbf{p} = A[x, \vartheta]p$*.*

2) Suppose that p *is a maximal ideal of the ring* A*; and let there exist a positive integer* n *such that* $x^n \in \mathbf{p}$*, but* $x^{n-1} \notin \mathbf{p}$*. Then*

$$\mathbf{p} = A[x, \vartheta]x^n + A[x, \vartheta]p$$

3) In the general case, if $x^n \in \mathbf{p}$ *for some positive integer* n*, then there exists* $a \in A - p$ *such that*

$$(\mathbf{p} : a) = A[x; \vartheta]x^m + A[x; \vartheta]p.$$

for some $1 \leq m \leq n$*.*

Proof. 1) If $\mathbf{p}$ does not contain x^n for any $n \geq 1$, then the ideal $\mathbf{p}$ is the preimage of a left ideal $\mathbf{p}'$ of the ring $A[x^{-1}, x, \vartheta]$; and $\mathbf{p}' \bigcap A = p$. So, the assertion follows from Lemma 1.3.2.

2)&3) Let now the ideal $\mathbf{p}$ contain x^n for some $n \geq 1$, but $x^{n-1} \notin \mathbf{p}$. Suppose that $\mathbf{p} \neq A[x, \vartheta]x^n + A[x, \vartheta]p$; and let

$$h(x) = x^i a_i + x^{i-1} a_{i-1} + ... + a_0, \quad a_i \neq 0,$$

be a nonzero polynomial from $\mathbf{p}$ of minimal degree with respect to the property: all the nonzero coefficients of h are from $A - p$.

For every $\lambda \in p$ we have:

$$\vartheta^i(\lambda)h(x) - h(x)\lambda = x^{i-1}(\vartheta(\lambda) - \lambda)a_{i-1} + ... + (\vartheta^i(\lambda) - \lambda)a_0$$

Since $\deg(\vartheta^i(\lambda)h(x) - h(x)\lambda) < \deg(h)$ and, for any m, there exists $\lambda \in p$ such that $\vartheta^m(\lambda) - \lambda \notin p$, all the coefficients $a_m, 0 \leq m \leq i-1$, are zeros; i.e. $h(x) = x^i a_i$.

Denote by p' the set of all elements $a \in A$ such that $x^i a \in \mathbf{p}$. It is easy to see that p' is an ideal in A. Note that the ideal p' is proper: otherwise the ideal $\mathbf{p}$ would contain x^i, which contradicts to the hypothesis about the minimality of the integer n such that $x^n \in \mathbf{p}$.

Obviously, p' contains p.

2) Therefore, if the ideal p is maximal, then $p' = p$ contradicting to the assumption.

3) Suppose now that the ideal p is not maximal, and p' is strictly greater then p. For any $a' \in p' - p$, the ideal $(\mathbf{p} : a')$ contains x^i, and, since p is prime,

$$(\mathbf{p} : a') \bigcap A = (p : a') = \ p.$$

Note that $i < n$. If $(\mathbf{p} : a')$ still does not coincide with $A[x; \vartheta]x^i + A[x; \vartheta]p$, we repeat the procedure and find an $a'' \in A - p$ such that $((\mathbf{p} : a') : a'') = (\mathbf{p} : a''a')$ contains x^ν for some $\nu < i$.

Clearly this process stabilizes and we shall come to the desired equality:

$$(\mathbf{p} : a) = A[x;\vartheta]x^m + A[x;\vartheta]p$$

for some $m < n$ and $a \in A - p$. ■

1.3.5. Proposition. *1) Let* $\mathbf{p}$ *be a left ideal from the left spectrum of* $A[x, x^{-1}, \vartheta]$ *and* $p = \mathbf{p} \bigcap A$ *is a prime ideal of the ring* A *which is not stable under the automorphism* ϑ^m *for any integer* m*. Then* $\mathbf{p} = A[x, x^{-1}, \vartheta](p \bigcap A)$*.*

2) Let p' *be a prime ideal of the ring* A*, which is not stable under the automorphism* ϑ^m *for any integer* m*. Then the left ideal* $\mathbf{p} = A[x, x^{-1}, \vartheta]p'$ *belongs to the left spectrum of the ring* $A[x, x^{-1}, \vartheta]$*.*

If the ideal p' *is maximal, then the left ideal* $A[x, x^{-1}, \vartheta]p'$ *is maximal.*

Proof. Let V denote the quotient $A[x, x^{-1}, \vartheta]$-module $A[x, x^{-1}, \vartheta]/\mathbf{p}$, and let V_0 be the image of the subring A in V. We have to check that, for any nonzero cyclic submodule M of the module V, there exists a diagram

$$(l)M \longleftarrow W \longrightarrow V$$

for some positive integer l with the left arrow being a monomorphism and the right arrow an epimorphism (cf. 3.10).

Let $M = A[x, x^{-1}, \vartheta] \cdot v$ for some nonzero element $v \in V$; and let $f(x) = \sum_{n' \leq i \leq n} x^i a_i$, $n, n' \geq 0$, be an element from the preimage of v such that each nonzero coefficient a_i of f belongs to $A - p'$. Clearly $n' \neq n$ (otherwise $x^{-n} f(x) = a_n$ is an element of p').

a) There exists an element s *of the ring* A *such that*

$$sx^{-n'} f(x) \in (A + \mathbf{p}) - \mathbf{p};$$

i.e. $sx^{-n'} f(x) = sa_{n'} + g(x)$, where $sa_{n'} \in A - p'$ and $g(x) \in \mathbf{p}$.

In fact, let $N = n - n'$ be the degree of the polynomial $h(x) := x^{-n'} f(x)$. By condition, there exists an element $t \in p'$ such that $\vartheta^N(t) \in A - p'$. We have:

$$th(x) = \vartheta^N(t)a_{n'} + x^N ta_n + xh_1(x)$$

Clearly $\vartheta^N(t)a_{n'} \in A - p'$, since $\vartheta^N(t)$ and $a_{n'}$ do not belong to p'; $x^N ta_n \in x^N p' \subset \mathbf{p}$; and $\deg(xh(x)) \leq N - 1$. Therefore we can proceed by induction.

b) Thus, applying to the image v of the element $f(x)$ the element $sx^{-n'}$ (cf. the heading *a)* of the proof), we obtain a nonzero element v' of the A-submodule V_0 of the module V. Since $V_0 \simeq A/p'$, where p' is a prime ideal of A, there exists a diagram

$$(l)Av' \longleftarrow W_0 \longrightarrow V_0 \tag{1}$$

for some positive integer l the left arrow of which is a monomorphism and the right one is an epimorphism. Note that the ring $A[x, x^{-1}, \vartheta]$ is flat over A; i.e. the functor $A[x, x^{-1}, \vartheta] \otimes_A$ is exact. Thus, to the diagram (1), there corresponds the diagram

$$(l)M = (l)A[x, x^{-1}, \vartheta] \cdot v \longleftarrow W = A[x, x^{-1}, \vartheta]W_0 \longrightarrow A[x, x^{-1}, \vartheta]V_0 = V,$$

we were looking for.

Suppose now that the ideal p' is maximal; i.e. the A-module $V_0 = A/p'$ is simple. Then the intersection of any cyclic submodule W with V_0, being nonzero, coincides with V_0; hence $W = V$. This means that $V = A[x, x^{-1}, \vartheta]/\mathbf{p}$, where $\mathbf{p} = A[x, x^{-1}, \vartheta]p'$, is a simple $A[x, x^{-1}, \vartheta]$-module; i.e. $\mathbf{p}$ is a left maximal ideal. ■

1.3.6. Lemma. *Let p be a prime ideal of the ring A such that $A[x, \vartheta]p \in Spec_l A[x, \vartheta]$. Then $\vartheta^n(p) \subseteq p$ for some $n \geq 1$.*

Proof. Consider the quotient $A[x, \vartheta]$-module $V = A[x, \vartheta]/A[x, \vartheta]p$. The module V is graded: $V = \oplus_{n \geq 0} V_n$, where $V_n \simeq A/\vartheta^n(p)$. Since V belongs to the spectrum of the category $A[x, \vartheta]$-mod (cf. Proposition 0.3), the submodule $V_+ := \oplus_{n \geq 1} V_n$ is a subquotient of the direct some of a finite number of copies of V. Since the forgetting functor $\mathcal{F} : A[x, \vartheta] - mod \longrightarrow A - mod$ is exact, this implies that $\mathcal{F}(V)$ is a subquotient of $\mathcal{F}(V_+)$. In particular, the support of the A-module $\mathcal{F}(V)$ is contained in the support of $\mathcal{F}(V_+)$. Since $Supp(\mathcal{F}(V_+)) = \bigcup_{n \geq 1} Supp(V_n)$ and $Supp(V_n)$ is the closure of $\vartheta^n(p)$ in the Zariski topology, $\vartheta^n(p) \subseteq p$ for some $n \geq 1$. ■

1.4. Using an algebra structure. Suppose that A is an algebra over a field k; and let ϑ be a k-algebra automorphism. We can gather some additional information taking into consideration the natural embedding $\varphi : k[x] \longrightarrow A[x, \vartheta]$.

Since φ is a left normal morphism (cf. I.3.2), the preimage $\mathbf{p} \bigcap k[x]$ of the ideal $\mathbf{p}$ belongs to $Spec k[x]$; i.e. $\mathbf{p} \bigcap k[x] = k[x]f$, where $f = f_p$ is either irreducible polynomial or zero (cf. Propositions 3.2.3).

Consider the case when $f \neq 0$; i.e. f is an irreducible polynomial. Since the nonzero coefficients of f do not belong to $\mathbf{p}$, it follows from Lemma 1.3.2 that there exists a positive integer $m \leq \deg(f)$ such that the ideal $\mathbf{p} \bigcap A$ is stable under the automorphism ϑ^m.

In particular, if the field k is algebraically closed, the intersection $\mathbf{p} \bigcap A$ *is stable under* ϑ itself.

Now, suppose that A is a domain, and let Q be the localization at the set of nonzero elements of A. Fix a nonzero prime (hence maximal) ideal $k[x]f$ of $k[x]$.

If $A[x, \vartheta]f \in Spec_l A[x, \vartheta]$, $Q(A[x, \vartheta]f) = K(A)[x, \vartheta]f$ belongs to $Spec_l K(A)[x, \vartheta]$ (because exact localizations respect the left spectrum). Therefore f is an irreducible element of the ring $K(A)[x, \vartheta]$.

Conversely, let a polynomial $f \in k[x]$ be an irreducible element of the ring $K(A)[x, \vartheta]$; and let $\mathbf{p}$ be an ideal from $Spec_l A[x, \vartheta]$, containing $A[x, \vartheta]f$. Since the ideal $A[x, \vartheta]f$ is the preimage of its localization – the maximal left ideal $K(A)[x, \vartheta]f$, either $\mathbf{p} = A[x, \vartheta]f$, or $\mathbf{p} \bigcap A \neq \{0\}$. In the second case, it follows from Proposition 1.3.4 that

there exists an element $a \in A - \mathbf{p}$ such that $(\mathbf{p} : \mathbf{a}) \bigcap A$ is a nonzero prime ideal stable under the automorphism ϑ^m for some integer m.

In particular, *if $f(0) \neq 0$, and, for any $m > 0$, there are no nonzero ϑ^m-stable ideals in $SpecA$, then the ideal $A[x, \vartheta]f$ is maximal.*

1.5. Example: the algebra $U_2(k, \alpha)$. Let $A = k[z]$, $u = z$; and let the automorphism ϑ is determined by the equality $\vartheta(z) = z + \alpha$; i.e. $A[x, \vartheta]$ is the ring $U_2(k, \alpha)$ generated by x, z with the relation

$$xz = zx + \alpha x \tag{1}$$

(cf. Example 1.1).

If $p \in Spec_l U_2(k,\alpha)$ and $p \bigcap k[z] \neq \{0\}$, then there exists an irreducible polynomial $h = h(y)$ such that $p \bigcap k[z] = k[z]h$. Invariance of the ideal $k[z]h$ with respect to ϑ^ν means that

$$\vartheta^\nu(h) = h(y + \nu\alpha) = u(y)h(y) \tag{2}$$

for some polynomial u. One can easily deduce from the equality $\deg(\vartheta^\nu(h)) = \deg(h)$ that $u = 1$; i.e. $h(y + \nu\alpha) = h(y)$. The last equality is possible only if $\deg(h) = 0$. Since the ideal $k[z]h$ is proper, h should be zero.

Assume that $char(k) = 0$. Then Propositions 1.3.4 and 1.3.6 provide the following description of $Spec_l U_2(k,\alpha)$.

a) There is the embedding

$$\lambda_z : Spec_l k[z] \longrightarrow Spec_l U_2(k,\alpha) \tag{3}$$

sending a prime ideal $k[z]h$ into the two-sided ideal $U_2(k,\alpha)x + k[z]h$ which is maximal iff $h \neq 0$.

b) There is an embedding

$$\gamma_x : Spec_l k[x] \longrightarrow Spec_l U_2(k,\alpha), \quad k[x]g \longmapsto U_2(k,\alpha)g. \tag{4}$$

If the polynomial g is not of the form cx, then $U_2(k,\alpha)g$ *is a maximal left ideal.* But it is not two-sided.

If $g = cx$, $c \in k^*$, then the set of specializations of $U_2(k,\alpha)g = U_2(k,\alpha)x$ coincides with the 'line'

$$\lambda_x(Spec_l k[z]) = \{U_2(k,\alpha)x + k[z]h \mid k[z]h \in Spec_l k[z]\}.$$

c) The remaining part of $Spec_l U_2(k,\alpha)$, denote it by $\Xi(U_2(k,\alpha))$, consists of the ideals p of the form

$$U_2(k,\alpha) \bigcap k(z)[x,\alpha]r, \tag{5}$$

where $k(z)[x,\alpha]$ is the localization of the algebra $U_2(k,\alpha)$ at $k[z] - \{0\}$, and $r = r(z,x)$ is a polynomial in (z,x) such that r is an irreducible element of the ring $k(z)[x,\alpha]$, but not of the form $f(z)g(x)$.

d) Finally, there is a generic point $\{0\}$. ∎

1.6. Remark. We could produce a similar analysis of the quantum plane. But, the quantum plane, besides being a generic skew polynomial ring over $k[y]$, has an additional advantage: it is a hyperblic ring (which is not the case with the algebra $U_2(k,\alpha)$ when $\alpha \neq 0$). The hyperbolic structure allows to get a description of the left spectrum of the quantum plane much more gracefully. We shall do it in Section 4. ∎

1.7. The remaining part of the spectrum. Now we return to a general skew polynomial ring $A[x,\vartheta]$ and a left ideal $\mathbf{p} \in Spec_l A[x,\vartheta]$ such that $p := \mathbf{p} \bigcap A$ is a prime ideal in A. It remains to consider the last among the listed in the section 1.3 alternatives:

(d) The ideal p is not ϑ -stable, but it is ϑ^n-stable for some $n \geq 2$.

A description of this part of the left spectrum in the full generality requires a more sophisticated technique. It is presented, among other things, in Chapter IV. Here (in the next section) we consider an important for applications special case; namely, we assume that $\vartheta^n = Id$.

2. The restricted skew polynomial rings.

2.1. Definition. Fix again a noetherian commutative ring A and an automorphism ϑ of A. Suppose that there exists an integer $n \geq 1$ such that $\vartheta^n = Id$. Finally, let u be a nonzero ϑ -invariant element of the ring $A:\ \vartheta(u) = u$.

Define the *n-restricted ϑ-skew polynomial ring*, $A[x; \vartheta \mid u, n]$, by the relations

$$xa = \vartheta(a)x \quad \text{for every} \quad a \in A, \quad x^n = u. \tag{1}$$

2.2. Example. Let ϑ be an automorphism of the ring A' such that $\vartheta^n = Id$ for some $n \geq 1$. Then y^n is a central element of the ring $A'[y, \vartheta]$; in particular, $A'[y^n]$ is a commutative subring of $A'[y, \vartheta]$. Set $A := A'[z]$. Denote by ϑ' the extension of the automorphism ϑ onto $A'[z]$ such that $\vartheta'(z) = z$. There is a natural isomorphism from $A'[y, \vartheta]$ onto $A[x; \vartheta \mid z, n]$ which sends a polynomial $f(y)$ into $f(x)$. ■

Now fix a restricted skew polynomial ring $A[x; \vartheta \mid u, n]$.

2.3. Lemma. *Every element of the ring $A[x; \vartheta \mid u, n]$ is uniquely represented as a polynomial $\sum_{0 \leq i < n} x^i a_i$ with coefficients in A.*

Proof. Note that $A[x; \vartheta \mid u, n]$ is the quotient of the ring $A[x; \vartheta]$ with respect to the two-sided ideal generated by $x^n - u$. Since x^n and u are both central elements of the ring $A[x, \vartheta]$, the generated by $x^n - u$ two- sided ideal coincides with the left ideal $A[x, \vartheta](x^n - u)$. This means that the canonical epimorphism maps a nonzero polynomial $h(x) \in A[x, \vartheta]$ into a zero element of the ring $A[x; \vartheta \mid u, n]$ if and only if $h(x) = f(x)(x^n - u)$ for some $f(x) \in A[x, \vartheta]$. In particular, either $\deg(h) \geq n$, or $h(x) \equiv 0$.

Clearly every element of the ring $A[x; \vartheta \mid u, n]$ is the image of a polynomial

$$g(x) = \sum_{0 \leq i < n} x^i a_i \in A[x, \vartheta],$$

and the argument above shows that the image of $g(x)$ is zero if and only if $g(x) \equiv 0$. ■

2.4. The decomposition. Being ϑ-invariant, u is a central element in $A[x; \vartheta \mid u, n]$. This implies that

$$Spec_l A[x; \vartheta \mid u, n] = V_l(u) \bigcup U_l(u).$$

Clearly

$$V_l(u) \simeq Spec_l A[x; \vartheta \mid 0, n] \simeq Spec A[x; \vartheta]/A[x; \vartheta]x \simeq Spec A,$$

since the ideal $J := A[x; \vartheta \mid 0, n]x$ is nilpotent: $J^n = \{0\}$.

As to the open subset $U_l(u)$, we have:

$$U_l(u) \simeq Spec_l A'[x; \vartheta' \mid u', n],$$

where $A' = (u)^{-1}A, u'$ is the image of u in A', ϑ' is the induced by ϑ automorphism of A'.

Since the element u' is invertible and $x^n = u'$, the element x is also invertible in $A'[x; \vartheta' \mid u', n]$. This means that $A'[x; \vartheta' \mid u', n] \simeq A'[x, x^{-1}; \vartheta' \mid u', n]$, where the ring on the right side is obtained from the ring $A'[x, x^{-1}; \vartheta']$ of skew Laurent polynomials by adding the relation $x^n = u'$.

2.5. Proposition. *Let* $\mathbf{p}$ *be a left ideal of the ring* $A[x; \vartheta \mid u, n]$ *such that* $p := \mathbf{p} \bigcap A$ *is a nonzero prime ideal in* A *which is not stable under the automorphism* ϑ^m *if* $1 \leq m < n$. *Suppose that the element* u *is invertible. Then* $\mathbf{p} = A[x; \vartheta \mid u, n]p$.

Proof. The assertion follows immediately from Corollary 1.3.3 and the preceeding observation (cf. the end of 2.4). ■

2.5.1. Remark. There is a straightforward analog of Proposition 1.3.4 for restricted skew polynomial rings. However, since this analog does not play any role in what follows (i.e. in the description of the left spectrum of hyperbolic rings), we leave it to the reader.

2.6. The rest of the spectrum. We shall follow the scenario outlined in 1.3.

(a) Suppose that the ring A is prime, and $\mathbf{p}$ is a left ideal from $Spec_l A[x; \vartheta \mid u, n]$ such that $\mathbf{p} \bigcap A = \{0\}$. Then $\mathbf{p}$ is the preimage of a left ideal, $\mathbf{p}\hat{}$, in the localized at the set $A - \{0\}$ ring $K(A)[x; \vartheta\hat{} \mid u\hat{}, n]$(cf. 1.3). But, the ring $K(A)[x; \vartheta\hat{} \mid u\hat{}, n]$ is a skew field. Therefore $\mathbf{p}\hat{} = \mathbf{0}$ which implies that $\mathbf{p} = \mathbf{0}$.

(b) Suppose now the left ideal $\mathbf{p} \in Spec_l A[x; \vartheta \mid u, n]$ is such that the intersection $p := \mathbf{p} \bigcap A$ is a ϑ -invariant prime ideal in A.

This case is reduced to the study of left ideals $\mathbf{p}'$ from $Spec_l A'[x; \vartheta' \mid u', n]$ such that $\mathbf{p}' \bigcap A' = \{0\}$ for the triple A', ϑ', u', where $A' = A/p$, ϑ' is the induced by ϑ automorphism, u' is the image of u: the ideal $\mathbf{p}$ is the preimage of such an ideal $\mathbf{p}'$ (cf. 1.3, *(b)*). This means that

either $\mathbf{p}$ is the ideal generated by x and p (the case when $u' \in p$);

or $\mathbf{p}$ is generated by p (when $u' \notin p$; cf. *(a)* above).

Note that in both cases $\mathbf{p}$ is a two-sided ideal.

If n is a prime number, the listed above cases exhaust all the possibilities. If n is not prime, there might be ϑ^m-stable, but not ϑ-stable, primes for an $m < n$.

We omit here the investigation of such cases. They will be cleared up in Chapter IV, where a complete description of the left spectrum is obtained for skew polynomial and hyperbolic rings over an arbitrary (noncommutative in general) "coefficient" ring.

3. The left spectrum and irreducible representations of hyperbolic rings.

To study simultaneously the universal enveloping algebra $U(sl(2,k))$ and different versions of quantum group $SL_q(2,k)$ (cf.[CK] and [MNSU]), as well as some other deformations of $U(sl(2,k))$ (for example, those from [S1]) the algebra $U_q(sl(2,k))$ is replaced by its straightforward generalization – the ring $A\langle\vartheta, u\rangle$ generated by a commutative ring A and by the indeterminates x, y, satisfying the relations:

$$xa = \vartheta(a)x, \quad ya = \vartheta^{-1}(a)y \quad \textit{for every } a \in A, \tag{1}$$

where ϑ is a fixed automorphism of the ring A, and

$$xy - yx = u \textit{ for some } u \in A;$$

In the case of $U_q(sl(2))$, $A = k[z, z^{-1}]$, $\vartheta f(z) = f(qz)$ for all $f \in k[z, z^{-1}]$; $u = (z - z^{-1})/(q - q^{-1})$.

In the case of $U(sl(2))$, $A = k[z]$, $u = z$, and $\vartheta f(z) = f(z + \alpha)$, $\alpha \in k^*$.

It follows from the relations (1) that the subring R generated by A and xy is commutative (actually, isomorphic to the polynomial ring $A[t]$). So, one can rewrite the relations in terms of R and elements x, y. This is the way the *hyperbolic* ring $R\{\theta, \xi\}$ appeared in the first place (in [R3]). Note that this class of algebras was introduced approximately at the same time (or earlier) by Bavula [Ba] under the name *generalized Weyl algebras.* Unfortunately, this became known to me, after [R] appeared.

In Section 3.1, we make a transition from the rings $A\langle\vartheta, u\rangle$ to hyperbolic rings and consider some motivating examples.

Section 3.2 contains the description of a part of the left spectrum of a hyperbolic ring which often happens to be the whole left spectrum (if the root of one or a base field of positive characteristic are not involved).

In Section 3.3, we introduce *restricted* hyperbolic rings which correspond to the "root of unity case" and show that the description of their left spectrum is reduced to the description of the left spectrum of some associated restricted skew polynomial rings.

3.1. Hyperbolic Rings.

3.1.0. The ring $A\langle\vartheta, u\rangle$. Let A be a commutative ring, ϑ its automorphism, u a fixed element of A. With this data, we relate the ring $A\langle\vartheta, u\rangle$ generated by the ring A and by the indeterminates x, y subject to the following relations:

$$xa = \vartheta(a)x, \quad ya = \vartheta^{-1}(a)y \quad \text{for any } a \in A; \tag{1}$$

$$xy - yx = u \quad \text{for some } u \in A; \tag{2}$$

3.1.1. Example. Let $A = k[z]$, $u = z$; and let the automorphism ϑ is determined by the equality: $\vartheta(z) = z + \alpha$. Then, obviously, $\vartheta^{-1}(z) = z - \alpha$; and $A\langle\vartheta, u\rangle$ turns out to be the k -algebra generated by x, y, z satisfying the relations

$$xz = zx + \alpha x, \quad yz = zy - \alpha y, \quad xy - yx = z \tag{3}$$

If $\alpha \neq 0$, then the relations (3) determine the universal enveloping algebra $U(sl(2, k))$ of the Lie algebra $sl(2, k)$. ∎

3.1.2. Example: the quantum universal enveloping algebras of sl(2,k). Now let $A = k[z, z^{-1}]$; and let

$$\vartheta(z) = qz, \quad u = (z^2 - z^{-2})/(q - q^{-1}), \tag{4}$$

where q is an element of $k - \{0, 1\}$. Then $A\langle\vartheta, u\rangle$ is the k-algebra generated by z, z^{-1}, x, y subject to the relations

$$zx = qxz, \quad zy = q^{-1}yz, \quad [x, y] = \frac{z^2 - z^{-2}}{q - q^{-1}} \tag{5}$$

This algebra is known as the *quantum universal enveloping algebra* $U_q(sl(2,k))$ *of* $sl(2,k)$.

Another version of quantum universal enveloping algebra of $sl(2,k)$ is obtained by taking $u = (z - z^{-1})/(q - q^{-1})$ [CK]. ∎

3.1.3. From the ring $A\langle\vartheta, u\rangle$ to the ring $A[\xi]\{\theta, \xi\}$. The defining the ring $A\langle\vartheta, u\rangle$ relations (cf. 3.1.0) show that the element $\xi = xy$ commutes with every element of the ring A; i.e. the ring $A[\xi]$, generated by A and ξ, is commutative. This fact suggests to consider $A\langle\vartheta, u\rangle$ not as an A-ring, but as an $A[\xi]$-ring.

Define the extensions θ and θ' of the automorphisms ϑ and ϑ^{-1} respectively onto $A[\xi]$, setting $\theta(\xi) = \zeta + \vartheta(u)$ and $\vartheta'(\xi) = \xi - u$. We have:

$$\theta \circ \theta'(\xi) = \theta(\xi - u) = (\xi + \vartheta(u)) - \vartheta(u) = \xi$$

and

$$\theta' \circ \theta(\xi) = \theta'(\xi + \vartheta(u)) = (\xi - u) + u = \xi.$$

In other words, $\theta' = \theta^{-1}$. Now the relations defining the ring $A\langle\vartheta, u\rangle$ (cf. 3.1.0) can be rewritten in the following way:

$$xb = \theta(b)x \quad \text{and} \quad yb = \theta^{-1}(b)y \quad \text{for all} \quad b \in A[\xi]; \tag{1}$$

$$xy = \xi, \quad yx = \theta^{-1}(\xi). \tag{2}$$

3.1.4. The hyperbolic ring $R\{\theta, \xi\}$. Let θ be an automorphism of a commutative ring R; and let ξ be an element of R. Denote by $R\{\theta, \xi\}$ the R-ring generated by the indeterminates x, y satisfying the relations:

$$xa = \theta(a)x \quad \text{and} \quad ya = \theta^{-1}(a)y \quad \text{for any } a \in R; \tag{1}$$

$$xy = \xi, \quad yx = \theta^{-1}(\xi). \tag{2}$$

Note that, under assumption that y is not a zero divisor, the second relation (2) follows from the first one and (1), since

$$(yx)y = y\xi = \theta^{-1}(\xi)y.$$

The ring $R\{\theta,\xi\}$ is called *hyperbolic* because of the relations (2) and (3) which can be interpreted as the equations of a (noncommutative) hyperbola.

3.1.5. Example: the coordinate algebra of $SL_q(2,k)$. The coordinate algebra $A(SL_q(2,k))$ of the *algebraic quantum group* $SL_q(2,k)$ (cf. [M]) is the k-algebra generated by the indeterminates x, y, u, v which satisfy the equations:

$$qux = xu, \quad qvx = xv, \quad qyu = uy, \quad qyv = vy, \quad uv = vu, \tag{1}$$

$$xy - quv = 1 = yx - q^{-1}uv \tag{2}$$

Now take the algebra $k[u,v]$ of polynomials in u, v as R, and set $\theta f(u,v) := f(qu, qv)$ for any polynomial $f(u,v)$. Finally, denote by ξ the element $1 + q^{-1}uv$ by ξ. Then the relations (1), (2) become equivalent to the relations (1), (2) in 3.1.4 determining the ring $R\{\theta,\xi\}$.■

3.1.6. Lemma. *Every element of the ring $R\{\theta,\xi\}$ can be represented as $f(x) + g(y)$, where*

$$f(x) = \sum_{m\geq 0} x^m a_m \quad \textit{and} \quad g(y) = \sum_{i\geq 1} y^i b_i$$

are uniquely determined polynomials with coefficients in R.

Proof. Clearly every element of $R\{\theta,\xi\}$ can be represented, thanks to the relations $xy = \xi$ and $yx = \theta^{-1}(\xi)$, as the sum of a polynomial in x and a polynomial in y.

The uniqueness follows from the fact that these relations define a multiplication on the direct sum

$$R[x,\theta] \oplus yR[y,\theta^{-1}] \tag{1}$$

and the obtained this way ring satisfies the relations (1)–(3) in 3.1.4. Therefore the obvious map

$$R[x,\theta] \oplus yR[y,\theta^{-1}] \longrightarrow R\{\theta,\xi\}$$

is a ring isomorphism. ■

3.1.7. Corollary. *Every nonzero left ideal of the ring $R\{\theta,\xi\}$ has a nonzero intersection either with $R[x,\vartheta]$ or with $R[y,\vartheta]$.*

Proof. Suppose that the left ideal m of the ring $R\{\theta,\xi\}$ contains an element $f(x) + g(y)$, where both $f(x)$ and $g(y)$ are nonzero polynomials; and let $\nu = \deg(g)$. Then $x^\nu(f(x) + g(y))$ is a nonzero polynomial in x. ■

3.1.8. The canonical anti-automorphism. It is easy to see that the formulas

$$\sigma(a) = \theta^{-1}(a) \quad \text{for any} \quad a \in R; \quad \sigma(x) = y, \quad \sigma(y) = x. \tag{1}$$

define an anti-automorphism of the ring $R\{\theta,\xi\}$.

3.1.9. The adjoint ring and the adjunction isomorphism. We call $R\{\theta^{-1},\theta^{-2}(\xi)\}$ *the adjunct to $R\{\theta,\xi\}$ ring.* One can check that the formulas

$$\Theta(a) = \theta^{-1}(a) \quad \text{for any } a \in R; \quad \Theta(x) = y, \quad \Theta(y) = x$$

define an isomorphism $\Theta : R\{\theta^{-1}, \theta^{-2}(\xi)\} \longrightarrow R\{\theta, \xi\}$. The inverse to Θ isomorphism is described, obviously, as follows:

$$\Theta^{-1}(a) = \theta(a) \text{ for any } a \in R; \quad \Theta(x) = y, \quad \Theta(y) = x$$

Thanks to the *adjunction* isomorphism Θ, we can, after finding half of the representations of $R\{\theta, \xi\}$, obtain the other half for free.

3.1.10. The hyperbolic rings and the rings $A\langle\vartheta, \rho, u\rangle$. Let R be a ring of polynomials with coefficients in the ring A: $R = A[t]$. Fix an automorphism θ of the ring R such that the subring A is invariant with respect to θ and consider the hypebolic ring $R\{\theta, t\}$.

It follows from the 'degree' considerations that

$$\theta(t) = at + b, \quad \theta^{-1}(t) = ct + d$$

for some $a, b, c, d \in A$. From the equalities

$$t = \theta \circ \theta^{-1}(t) = \theta(ct + d) = \theta(c)(at + b) + \theta(d) =$$

$$\theta(c)at + \theta(c)b + \theta(d),$$

$$t = \theta^{-1} \circ \theta(t) = \theta^{-1}(at + b) = \theta^{-1}(a)(ct + d) + \theta^{-1}(b) =$$

$$\theta^{-1}(a)ct + \theta^{-1}(a)d + \theta^{-1}(b)$$

we obtain:

$$c = \theta^{-1}(a^{-1}); \quad d = -\theta^{-1}(\theta(c)b) = -c\theta^{-1}(b) = -\theta^{-1}(a^{-1}b) \tag{1}$$

Futher, it follows from (1) and from the equality (4) in 3.1.4 that (since by definition $xy = t$)

$$yx = \theta^{-1}(t) = ct + d = \theta^{-1}(a^{-1})[t - \theta^{-1}(b)]. \tag{2}$$

The equations (2) and $xy = t$ imply that

$$xy - \theta^{-1}(a)yx = \theta^{-1}(b) \tag{3}$$

On the other hand, for a given ring A and its automorphism ϑ, consider the ring $A\langle\vartheta; \rho, u\rangle$ generated by the indeterminates x, y satisfying the relations

$$xa = \vartheta(a)x, \quad ya = \vartheta^{-1}(a)y \quad \text{for any } a \in R; \tag{4}$$

$$xy - \rho yx = u. \tag{5}$$

where ρ is an invertible and u is an arbitrary element of A.

Now set $t = xy$. It is easy to see that the element t commutes with any element a of A. One can also verify that the ring generated by A and t is isomorphic to the ring $A[t]$ of polynomials in t with coefficients in A.

Define an extension of the automorphism ϑ up to endomorphism θ of the ring $A[t]$ as follows:

$$\theta(t) = \vartheta(\rho)t + \vartheta(u). \tag{6}$$

One can easily check that the formulas

$$\theta'(t) = \rho^{-1}(t - u), \quad \theta'(a) = /vartheta^{-1}(a) \tag{7}$$

for any $a \in A$ determine an inverse to θ endomorphism of $A[t]$.

In fact,

$$\theta \circ \theta'(t) = \theta(\rho^{-1}(t-u)) = \vartheta(\rho)^{-1}(\theta(t) - \vartheta(u)) = \vartheta(\rho)^{-1}\vartheta(\rho)t = t.$$

Similarly, $\theta' \circ \theta(t) = t$.

Thus, the hyperbolic rings $A[t]\{\theta, t\}$, where θ runs over the set of all the automorphisms of the ring $A[t]$ under which A is invariant, are in one-one correspondence with rings $A\langle\vartheta, \rho, u\rangle$, where ϑ is an automorphism of A, ρ is an invertible element and u is an arbitrary element of A. ■

3.1.10.1. Note. It follows from the equation (3) that the ring $A[t]\{\theta, t\}$ coincides with the ring $A\langle\vartheta, u\rangle$ (cf. 3.1.0) for an appropriate ϑ and u (ϑ is the induced by θ automorphism of the ring A, $u = \theta^{-1}(b)$; cf. (3)) if and only if $\theta(t) = t + b$ for some $b \in A$. ■

3.2. The left spectrum of a hyperbolic ring.

3.2.1. From the prime spectrum of R to the left spectrum of $R\{\theta, \xi\}$. In this section, we assume for convenience that the ring R is noetherian. By Lemma I.3.4.1 (and the following the lemma short discussion), this guarantees that, for any $\langle\mathbf{p}\rangle \in \mathbf{Spec}_l R\{\theta, \xi\} := Spec_l R\{\theta, \xi\}/\approx$, the subset

$$^a i(\langle\mathbf{p}\rangle) := \{p \in SpecR \mid \mathbf{p}' \bigcap R = p \quad \text{for some } \mathbf{p}' \in \langle\mathbf{p}\rangle\} \tag{1}$$

of $SpecR$ is nonempty. Here i is the embedding $R \longrightarrow R\{\theta, \xi\}$.

Actually, this is the only place, where the noetherian hypothesis is used. The results of this Section are valid for any associative ring R provided that only those points $\mathbf{p}$ of $\mathbf{Spec}_l R\{\theta, \xi\}$ are considered for which the set $^a i(\langle\mathbf{p}\rangle)$ is nonempty.

The problems which occupy this section are:

(a) to describe the correspondence $\langle\mathbf{p}\rangle \longmapsto {}^a i(\langle\mathbf{p}\rangle)$;

(b) to find (if possible) the inverse to $^a i$ map.

Consider the set of orbits, $SpecR/(\theta)$, of the action of the group $(\theta) := \{\theta^n \mid n \in \mathbb{Z}\}$ on $SpecR$. Denote by $SpecR/(\theta)_\xi$ the set of orbits $\Omega \in SpecR/(\theta)$ such that $\xi \notin p$ for any $p \in \Omega$. And let $Spec(R \mid \theta, \xi)$ be the preimage in $SpecR$ of the complement to $SpecR/(\theta)_\xi$. If the set $^a i(\langle\mathbf{p}\rangle)$ intersects with $Spec(R \mid \theta, \xi)$, then it lies entirely inside of $Spec(R \mid \theta, \xi)$.

Theorem 3.2.2 provides the solution of both problems for those $\langle\mathbf{p}\rangle$ which land in $Spec(R \mid \theta, \xi)$ in the case of infinite orbits.

Proposition 3.2.3 establishes that each infinite orbit from $SpecR/(\theta)_\xi$ is the set $^a i(\langle\mathbf{p}\rangle)$ for a unique $\langle\mathbf{p}\rangle$, and the map which assigns to a prime ideal p the left ideal $R\{\theta, \xi\}p$ of the ring $R\{\theta,\xi\}$ induces a bijection of the set $SpecR/(\theta)_{\xi,\infty}$ of infinite orbits onto the corresponding part of $\mathbf{Spec}_l R\{\theta,\xi\}$.

Generic finite orbits require more sophisticated technique. They are studied in Chapter IV. Instead, we consider several important special cases.

3.2.2. Theorem. *(i) Let $p \in SpecR$, and let the orbit of p is infinite.*

0) If $\theta^{-1}(\xi) \in p$, and $\xi \in p$, then the left ideal

$$p_{1,1} := p + R\{\theta,\xi\}x + R\{\theta,\xi\}y$$

is a two-sided ideal from $Spec_l R\{\theta,\xi\}$.

1) If $\theta^{-1}(\xi) \in p, \theta^i(\xi) \notin p$ for $0 \le i \le n-1$, and $\theta^n(\xi) \in p$, then the left ideal

$$p_{1,n+1} = R\{\theta,\xi\}p + R\{\theta,\xi\}x + R\{\theta,\xi\}y^{n+1}$$

in the ring $R\{\theta,\xi\}$ belongs to $Spec_l R\{\theta,\xi\}$.

2) If $\theta^i(\xi) \notin p$ for $i \ge 0$ and $\theta^{-1}(\xi) \in p$, then

$$p_{1,\infty} = R\{\theta,\xi\}p + R\{\theta,\xi\}x$$

belongs to $Spec_l R\{\theta,\xi\}$.

3) If $\xi \in p$ and $\theta^{-i}(\xi) \notin p$ for $i \ge 1$, then the left ideal

$$p_{\infty,1} = R\{\theta,\xi\}p + R\{\theta,\xi\}y$$

belongs to $Spec_l R\{\theta,\xi\}$.

(ii) If the ideal p in 1), 2) or 3) is maximal, then the corresponding left ideal of $Spec_l R\{\theta,\xi\}$ is maximal.

(iii) Every ideal $\mathbf{p}$ of $Spec_l R\{\theta,\xi\}$ such that $\theta^\nu(\xi) \in \mathbf{p}$ for a$\nu \in \mathbb{Z}$ is equivalent to one of them for a uniquely defined $p \in SpecR$. The latter means that

if p and p' are prime ideals of the ring R and (α,β) and (ν,μ) take values $(1,\infty)$, $(\infty,1)$, (∞,∞), or $(1,n)$, then $p_{\alpha,\beta}$ is equivalent to $p'_{\nu,\mu}$ if and only if $\alpha=\nu$, $\beta=\mu$, and $p=p'$.

Proof. (i) Consider the cyclic modules corresponding to the ideals. Let m be one of the ideals from the list. We shall prove that m belongs to the left spectrum of $R\{\theta,\xi\}$ by showing that, for any cyclic (nonzero) submodule W of R'/m there is a diagram of module morphisms

$$(l)W \longleftarrow N \longrightarrow R\{\theta,\xi\}/m, \qquad (l)$$

where the right arrow is a monomorphism and the left one is an epimorphism.

Take a nonzero element v of the module $R\{\theta,\xi\}/m$.

a) Suppose first that $v \in V_0 \simeq R/p$. Since the ideal p is prime, the cyclic R-submodule Rv is isomorphic to V_0. This implies that the cyclic submodule $R\{\theta,\xi\}v$ is isomorphic to $R\{\theta,\xi\}/m$.

Note that the assertion *0)* is already proved, since $R\{\theta,\xi\}/m$ coincides with its zero component V_0.

b) It is clear now that, if $v \in R\{\theta, \xi\}/m - V_0$, it suffices to show that the cyclic module $R\{\theta, \xi\}v$ contains a nonzero element from V_0.

Let

$$f(x) + g(y) = \sum_{0 \le i \le s} x^i a_i + \sum_{0 \le j \le \nu} y^j b_j$$

be a preimage of v in $R\{\theta, \xi\}$ such that $a_s \notin p$ and $b_\nu \notin p$, and, necessarily, $s + \nu \ge 1$. Then

1) In the first case, $1 \le s \le n,\ \nu = 0$, and

$$y^s(f(x) + g(y)) \in a_s \prod_{1 \le d \le s-1} \theta^d(\xi) + p_{1,n}.$$

Since a_s and $\theta^d(\xi)$, $1 \le d \le s-1$, belong to $R - p$, the element $y^s v$ is nonzero and belongs to $V_0 \simeq R/p$.

2) In the second case, $s \ge 1,\ \nu = 0$; and, as above, $y^s v$ is a nonzero element of $V_0 \simeq R/p$.

3) In the third case, $x^\nu v$ is a nonzero element of V_0.

(ii) According to *(i)*, every nonzero submodule, W, of $R\{\theta, \xi\}/m$ has a nonzero intersection with the R-submodule $V_0 = R/p$. If the ideal p is maximal, then V_0 is a simple R-module; hence W contains V_0 which implies that $W = R\{\theta, \xi\}/m$. Thus, $R\{\theta, \xi\}/m$ is a simple $R\{\theta, \xi\}$-module, or, equivalently, m is a left maximal ideal.

(iii) Let $\mathbf{p}$ be a left ideal from $Spec_l R\{\theta, \xi\}$ such that $\theta^\nu(\xi) \in \mathbf{p}$ for some integer ν.

(a) We claim that *in that case the ideal* $\mathbf{p}$ *is equivalent to an ideal* $\mathbf{p}' \in Spec_l R\{\theta, \xi\}$ *which contains either* x*, or* y*.*

It suffices to prove the assertion for $\nu \ge 0$, because the case of negative ν is obtained by dualization (i.e. by switching to the adjoint hyperbolic ring, cf. 3.2.7).

Consider the alternatives.

(a1) $y^{\nu+1} \notin \mathbf{p}$. Then the left ideal $(\mathbf{p} : y^{\nu+1})$ is equivalent to $\mathbf{p}$, and it contains x, since

$$xy^{\nu+1} = \xi y^\nu = y^\nu \theta^\nu(\xi) \in \mathbf{p}.$$

(a2) $y^{\nu+1} \in \mathbf{p}$. Then there is $n \ge 1$ such that $y^n \in \mathbf{p}$, and $y^{n-1} \notin \mathbf{p}$. Thus, the ideal $(\mathbf{p} : y^{n-1})$ is equivalent to $\mathbf{p}$, and it contains y.

(b) Thus, we can assume that $\mathbf{p}$ contains either x, or y. Consider the case $y \in \mathbf{p}$.

(b1) $y \in \mathbf{p}$. Since R is a noetherian ring, there exists an $r \in R$ such that $p := (\mathbf{p} : r) \bigcap R \in SpecR$. Clearly $y \in (\mathbf{p} : r)$.

(b1.0) If $x^n \notin (\mathbf{p} : r)$ for any $n \ge 1$, then, by Proposition 2.3.4, the intersection $(\mathbf{p} : r) \bigcap R[x, \theta]$ coincides with $R[x, \theta]p$. It follows from the assertion 2) of Lemma 3.2.4 that

$$(\mathbf{p} : r) = (\mathbf{p} : r) \bigcap R[x, \theta] + (\mathbf{p} : r) \bigcap R[y, \theta] = R[x, \theta]p + R[x, \theta]y;$$

i.e. $(\mathbf{p} : r) = p_{\infty,1}$.

(b1.1) If some power of x belongs to $(\mathbf{p} : r)$, then, by the assertion *3)* of Proposition 2.3.4, there exists an element a of R such that

$$((\mathbf{p} : r) : a) \bigcap R[x, \theta] = (\mathbf{p} : ar) \bigcap R[x, \theta] = R[x, \theta]p + R[x, \theta]x^m$$

for some positive integer m. Since

$$yx^m = (yx)x^{m-1} = \theta^{-1}(\xi)x^{m-1} = x^{m-1}\theta^{-m}(\xi),$$

this implies that $\theta^{-m}(\xi) \in p$. Let n be the maximal integer between *0* and $m-1$ such that $\theta^{-n}(\xi) \in p$

(b1.1.0) Let $n = 0$. Then

$$(\mathbf{p} : ar) = (p : ar) \bigcap R[y, \theta] + (\mathbf{p} : ar) \bigcap R[x, \theta] =$$

$$R\{\theta, \xi\}p + R\{\theta, \xi\}y + R\{\theta, \xi\}x^m.$$

One can see that the left ideal

$$((\mathbf{p} : ar) : x^{m-1}) = (\mathbf{p} : x^{m-1}ar)$$

is equivalent to $\mathbf{p}$, and contains x and y^m, since

$$y^m x^{m-1} = (\prod_{1 \le i \le m-1} \theta^i(\xi))y \in (\mathbf{p} : ar).$$

Note that $y^{m-1} \notin (\mathbf{p} : x^{m-1}ar)$. In fact,

$$y^{m-1}x^{m-1} = \prod_{1 \le i \le m-1} \theta^{-i}(\xi) \notin p,$$

since, by assumption, $\theta^{-i}(\xi) \notin p$ if $0 \le i \le m-1$, and the ideal p is prime.

Note also that $(\mathbf{p} : x^{m-1}ar) \bigcap R = \theta^{m-1}(p)$.

Indeed, set $p' := (\mathbf{p} : x^{m-1}ar) \bigcap R$. Clearly, $\theta^{m-1}(p) \subseteq p'$. By the same reason

$$\theta^{1-m}(p') \subseteq ((\mathbf{p} : x^{m-1}ar) : y^{m-1}) \bigcap R = ((\mathbf{p} : ar) : y^{m-1}x^{m-1}) \bigcap R = p.$$

Thus, $p \subseteq \theta^{1-m}(p') \subseteq p$ which means that $p' = \theta^{m-1}(p)$.

We have showed that

$$(\mathbf{p} : x^{m-1}ar) = p'_{1,m} = R\{\theta, \xi\}p' + R\{\theta, \xi\}x + R\{\theta, \xi\}y^m$$

is an ideal from $Spec_l R\{\theta, \xi\}$.

(b1.1.1) Suppose now that $1 \le n \le m-1$. Clearly, the left ideal

$$((\mathbf{p} : ar) : x^n) = (\mathbf{p} : x^n ar)$$

is equivalent to $\mathbf{p}$, and contains both y and x^s, where $s = m - n$. There exists $\lambda \in R$ such that $p' := ((\mathbf{p} : x^n ar) : \lambda) = (\mathbf{p} : \lambda x^n ar) \bigcap R$ is a prime ideal in R.

If $\theta^{-i}(\xi) \in (\mathbf{p} : \lambda x^n ar)$ for some $1 \le i \le s-1$, we repeat the procedure. This way, we shall come to the case $(b1.1.0)$ above.

(b2) If $x \in \mathbf{p}$, then a part of the argument above shows that either $\mathbf{p} \approx p_{1,\infty}$, or $\mathbf{p} \approx p_{1,\nu}$ for some $\nu \geq 1$.

(c) It remains to show the uniqueness:

In the representation $R\{\theta,\xi\}/p_{1,n}$ both elements x and y annihilate some nonzero elements, while in the representations $R\{\theta,\xi\}/p'_{1,\infty}, R\{\theta,\xi\}/p'_{\infty,1}$ and $R\{\theta,\xi\}/p'_{\infty,\infty}$ respectively y, x and both act injectively.

Thus, if $p_{1,n} \leq p'_{\nu,\mu}$, then $\quad \nu, \mu = 1, m \quad$ for some m.

Note that $n \geq m$.

In fact, if $n < m$, then y^{n+1} annihilates the module $R\{\theta,\xi\}/p_{1,n}$ and does not annihilate $R\{\theta,\xi\}/p'_{1,m}$; i.e.

$$(p_{1,n} : R\{\theta,\xi\}) \ni y^{n+1} \notin (p'_{1,m} : R\{\theta,\xi\}) \tag{1}$$

But, the relation $p_{1,n} \leq p'_{1,m}$ implies that the inclusion

$$(p_{1,n} : R\{\theta,\xi\}) \subseteq (p'_{1,m} : R\{\theta,\xi\})$$

which contradicts to (1). Thus, $n \geq m$. In particular, if $p'_{1,m}$ is equivalent to $p_{1,n}$, then $n = m$.

(c1) The relation $p_{1,n} \leq p'_{1,m}$ means that there is a diagram of $R\{\theta,\xi\}$-modules

$$\mathbb{V} := (\nu)R\{\theta,\xi\}/p_{1,n} \xleftarrow{\ \mathfrak{i}\ } K \xrightarrow{\ \mathfrak{e}\ } \mathbb{V}' := R\{\theta,\xi\}/p'_{1,m}, \tag{2}$$

where i is a monomorphism, and e is an epimorphism. The module $(\nu)R\{\theta,\xi\}/p_{1,n}$ can be written as $\bigoplus_{0\leq i\leq n} y^i((\nu)V)$, where $V = R/p$. In particular, it is isomorphic, as an R-module, to $\bigoplus_{0\leq i\leq n}(\nu)R/\theta^{-i}(p)$.

Similarly,

$$\mathbb{V}' := R\{\theta,\xi\}/p'_{1,m} = \oplus_{0\leq i\leq m} y^i V',$$

where $V' = R/p'$.

Thus, the diagram (2) induces the diagram

$$\oplus_{0\leq i\leq n}(\nu)R/\theta^{-i}(p) \xleftarrow{\ \mathfrak{i}'\ } K_0 \xrightarrow{\ \mathfrak{e}'\ } R/p', \tag{3}$$

where $K_0 := \mathfrak{e}^{-1}(V')$ and $\mathfrak{i}'$ is the restriction of $\mathfrak{i}$ to K_0.

The diagram (3) implies that

$$p' \in Supp(\oplus_{0\leq i\leq n}(\nu)R/\theta^{-i}(p)) = \bigcup_{0\leq i\leq n} Supp(R/\theta^{-i}(p));$$

i.e. $\theta^{-i}(p) \subseteq p'$ for some $1 \leq i \leq n$.

If $\mathbb{V} \approx \mathbb{V}'$, then $n = m$, and $p \subseteq \theta^i(p') \subseteq \theta^{i+j}(p)$, where i, j take values 0 or n. Since the ring R is noetherian (in particular, p has a finite height), the inclusion $p \subseteq \theta^{i+j}(p)$ implies that $p = \theta^{i+j}(p)$. Hence $p = \theta^i(p') = \theta^{i+j}(p)$.

Since $\theta^{-1}(\xi) \in p'$, the equality $p = \theta^i(p')$ implies that $\theta^{i-1}(\xi) \in p$. Since $0 \leq i \leq n$, and $\theta^j(\xi) \notin p$ if $0 \leq j \leq n-1$, the only remaining possibility is $i = 0$; i.e. $p' = p$.

(c2) Let now $p_{1,\infty} \approx p'_{1.\infty}$. Then the same argument, as in $(c1)$ shows that

$$p = \theta^i(p') = \theta^{i+j}(p) \text{ for some } i, j \geq 0.$$

This implies that $\theta^{i-1}(\xi) \in p$ which means (since $\theta^j(\xi) \notin p$ for $j \geq 0$) that $i = 0$; i.e. again $p = p'$.

(c3) The implication $p_{\infty,1} \approx p'_{\infty,1} \Rightarrow p = p'$ follows from (c2) by switching to the adjoint hyperbolic ring. ■

3.2.2.1. Remark. It is easy to describe the set ${}^a i(\langle \mathbf{p} \rangle)$ (cf. 3.2.1), if $\mathbf{p}$ is from the list of Theorem 3.2.2:

$${}^a i(\langle p_{1,1} \rangle) = \{p\}, \quad {}^a i(\langle p_{1,n} \rangle) = \{\theta^{-l}(p) \mid 0 \leq l \leq n-1\};$$

$${}^a i(\langle p_{1,\infty} \rangle) = \{\theta^{-l}(p) \mid l \geq 0\}, \quad {}^a i(\langle p_{\infty,1} \rangle) = \{\theta^{l}(p) \mid l \geq 1\}.$$

It might be useful to specify the "inverse" to ${}^a i$ map from the set $Spec(R \mid \theta, \xi)$ (which consists of all $p \in SpecR$ such that $\xi \in \theta^n(p)$ for some n; cf. 3.2.1) into $Spec_l R\{\theta, \xi\}$. This map, χ, is defined as follows:

a) If $\xi \in p \bigcap \theta(p)$, then $\chi(p) = p_{1,1}$.

b) If $\xi \in \theta^n(p) \bigcap \theta^m(p) - \bigcup_{n<i<m} \theta^i(p)$ for some $n \leq 0 \leq m$ such that $m - n \geq 2$, then $\chi(p) = \theta^{m-1}(p)_{1,m-n}$.

c) If $\xi \in \theta^n(p) - \bigcup_{n<i<\infty} \theta^i(p)$ for some $n \leq 0$, then $\chi(p) = \theta^n(p)_{\infty,1}$.

d) If $\xi \in \theta^m(p) - \bigcup_{-\infty<i<m} \theta^i(p)$ for some $m \geq 1$, then $\chi(p) = \theta^{m-1}(p)_{1,\infty}$.

Note that these numbers, m and n, are uniquely defined in each case which implies that χ is well defined. ■

3.2.3. Proposition. *(i) Let p be a prime ideal of the ring R such that $\theta^i(\xi) \notin p$ and $\theta^i(p) - p \neq \emptyset$ for every integer i.*

Then the ideal $p_{\infty,\infty} := R\{\theta, \xi\}p$ belongs to $Spec_l R\{\theta, \xi\}$.

(ii) Moreover, if $\mathbf{p}$ is a left ideal in $R\{\theta, \xi\}$ such that $\mathbf{p} \bigcap R = p$, then $\mathbf{p} = p_{\infty,\infty}$.

In particular, if p is a maximal ideal, then $p_{\infty,\infty}$ is a maximal left ideal.

(iii) If a prime ideal p' in R is such that $p_{\infty,\infty} \approx p'_{\infty,\infty}$, then $p' = \theta^n(p)$ for some integer n.

Conversely, $\theta^n(p)_{\infty,\infty} \approx p_{\infty,\infty}$ for every $n \in \mathbb{Z}$.

Proof. *(i)* As in the proof of Theorem 3.2.2, it is enough to show that any nonzero cyclic submodule $R\{\theta, \xi\}v$ of $R\{\theta, \xi\}/p_{\infty,\infty}$ contains a nonzero element from $V_0 \simeq R/p$.

Let

$$f(x) + g(y) = \sum_{0 \leq i \leq s} x^i a_i + \sum_{0 \leq j \leq \nu} y^j b_j$$

be a preimage of v in $R\{\theta, \xi\}$ such that $a_s \notin p$ and $b_\nu \notin p$. Multiplying by x^η for an appropriate η, $\eta \geq \nu$, we can assume that $g(y) = 0$ and $a_0 \notin p$. Now we can proceed by induction.

The case $s = 0$ is trivial.

If $s \geq 1$, there exists (by condition on θ and p) an element $r \in p$ such that $\theta^s(r) \notin p$. We have

$$\theta^s(r)f(x) = x^s a_s r + f'(x) \in f'(x) + R\{\theta, \xi\}p,$$

where $deg f' \leq s-1$ and $f'(0) = \theta^s(r)a_0 \notin p$.

(ii) Let $\mathbf{p}$ be a left ideal in the ring $R\{\theta, \xi\}$ such that $\mathbf{p} \bigcap R = p$. Clearly $\mathbf{p} \supseteq p_{\infty,\infty} := R\{\theta, \xi\}p$. Suppose that $\mathbf{p} \neq \mathbf{p}_{\infty,\infty}$; i.e. $\mathbf{p}$ contains a nonzero polynomial $f(x) + g(y)$ with all nonzero coefficients from $R - p$. Consider the alternatives:

a) $g(y) = 0$.

b) $f(x) = 0$. It follows from the fact that p is prime and $\vartheta^i(\xi) \notin p$ for all i that $x^\nu g(y)$, where $\nu = \deg(g)$, is a nonzero polynomial in x with all nonzero coefficients from $R - p$.

c) $f(x) + g(y) = \sum_{0 \leq i \leq s} x^i a_i + \sum_{1 \leq j \leq \nu} y^j b_j$, where $a_s \notin p$ and $b_\nu \notin p$. But then $x^\nu(f(x) + g(y))$ is a nonzero polynomial in x with all nonzero coefficients from $R - p$.

So that if $\mathbf{p} \neq p_{\infty,\infty}$, then $\mathbf{p}_x := \mathbf{p} \bigcap R[x;\theta]$ contains a nonzero polynomial with nonzero coefficients from $R - p$. This implies, by Proposition 2.4, that there exists $a \in R - p$ such that

$$(\mathbf{p}_x : a) = (\mathbf{p} : a) \bigcap R[x;\theta] = R[x;\theta]x^n + R[x;\theta]p$$

for some $n \geq 1$. Since $yx^n = \theta^{-1}(\xi)x^{n-1} = x^{n-1}\theta^{-n}(\xi)$, it follows from the last equality that $\theta^{-n}(\xi) \in p$. But, this contradicts to the assumption of this Proposition that $\theta^i(\xi) \notin p$ for any i.

(iii) Fix a positive integer n. Since $x^n \notin p_{\infty,\infty}$, the left ideal $(p_{\infty,\infty} : x^n)$ is equivalent to $p_{\infty,\infty}$.

Clearly

$$\theta^n(p) \subseteq p' := (p_{\infty,\infty} : x^n) \bigcap R;$$

hence

$$p \subseteq \theta^{-n}(p') \subseteq ((p_{\infty,\infty} : x^n) : y^n) \bigcap R = ((p_{\infty,\infty} : y^n x^n) \bigcap R.$$

But,

$$y^n x^n = \prod_{1 \leq i \leq n} \theta^{-i}(\xi) \in R - p$$

which implies the equality

$$((p_{\infty,\infty} : y^n x^n) = p_{\infty,\infty}.$$

In particular,

$$((p_{\infty,\infty} : y^n x^n) \bigcap R = \ p.$$

All together shows that $p' = \theta^n(p)$, and $(p_{\infty,\infty} : x^n) = p'_{\infty,\infty}$.

Dually, the ideal $(p_{\infty,\infty} : y^n) = \theta^{-n}(p)_{\infty,\infty}$ is equivalent to $p_{\infty,\infty}$.

Let now p' be another prime ideal in R.

The argument similar to that of the part (c1) of the proof of Theorem 3.2.2 shows that the relation $p_{\infty,\infty} \leq p'_{\infty,\infty}$ implies that $p \subseteq \theta^n(p')$ for some $n \in \mathbb{Z}$. Thus, if $p_{\infty,\infty} \approx p'_{\infty,\infty}$, then $p \subseteq \theta^n(p') \subseteq \theta^m(p)$ which, thanks to the noetherian property of R, implies that p is equal to the ideal $\theta^n(p')$. ∎

3.2.4. The Generating function. Following the tradition, we can concentrate all the information about the equivalence classes of ideals from $Spec_l R\{\theta,\xi\}$, which have a nonzero intersection with R, in one formal power series in λ and λ^{-1},

$$\mathfrak{G}(\lambda;\theta,\xi) := \sum_{i\in\mathbb{Z}} \theta^i(\xi)\lambda^i, \qquad (1)$$

which we call *generating function* of the ring $R\{\theta,\xi\}$.

3.2.5. The 'independent' part of the left spectrum. Assume that R is prime and consider those $\mathbf{p} \in Spec_l R\{\theta,\xi\}$ for which $\mathbf{p} \bigcap R = \{0\}$.

Note that if S is a multiplicative subset in R which is θ-invariant, then S is an Ore subset in $R\{\theta,\xi\}$. In particular, $R - \{0\}$ is an Ore subset in $R\{\theta,\xi\}$. So that we can localize the ring $R\{\theta,\xi\}$ at the multiplicative set $R - \{0\}$ and obtain as a result the ring $K(R)\{\Theta,\xi'\}$, where $K(R)$ is the field of fractions of R, Θ the induced by θ automorphism of $K(R)$, ξ' the image of ξ in $K(R)$. Since localizations respect the left spectrum, the localization Q at the set $R - \{0\}$ sends the ideal $\mathbf{p}$ into the left ideal $Q\mathbf{p}$ from $Spec_l K(R)\{\Theta,\xi'\}$.

Note now that the element ξ, being nonzero, is invertible in the ring $K(R)$; and the relation $yx = \xi$ means that $y = \xi x^{-1}$. Therefore the ring $K(R)\{\Theta,\xi'\}$ is isomorphic to the ring $K(R)[x,\Theta]$. In particular, the ideal $Q\mathbf{p}$ is determined by an irreducible element $r = r(x)$ of the ring $K(R)[x,\Theta]$: $\quad Q\mathbf{p} = K(R)\{\theta,\xi\}r$.

Note that the localization Q sends the (skew polynomial) subring $R[x,\vartheta]$ generated by R and x into the subring $K(R)[x,\vartheta]$ of the ring $K(R)\{\theta,\xi\}$.

3.2.6. Points over θ-invariant prime ideals. Suppose now that $\mathbf{p}$ is a left ideal from $Spec_l R\{\theta,\xi\}$ such that $p = \mathbf{p}\bigcap R$ is a θ-invariant prime ideal in R. Then θ induces an automorphism, θ', of the quotient ring $R' = R/p$, and the canonical map $\pi : R \longrightarrow R'$ extends to a ring morphism

$$\pi' : R\{\theta,\xi\} \longrightarrow R'\{\theta',\xi'\},$$

where $\xi' = \pi(\xi)$, $\quad \pi'(x) = x$, $\quad \pi'(y) = y$. Since π' is an epimorphism, the image $\mathbf{p}'$ of the ideal $\mathbf{p}$ belongs to the left spectrum, and $\mathbf{p}' \bigcap R' = \{0\}$.

There are two possibilities: either $\xi \in p$, or $\xi \notin p$.

Consider each of them.

(a) Degenerate case: $\xi \in p$. This implies that, since p is θ-invariant, $\theta^{-1}(\xi) \in p$. Thus, both xy and yx are in p. This means that the ring $R'\{\theta',\xi'\} = R'\{\theta',0\}$ is defined by the relations:

$$xr = \theta'(r)x, \quad ry = y\theta'(r) \quad \text{for any } r \in R',$$

$$xy = 0 = yx.$$

We shall write $R'\{\theta'\}$ instead of $R'\{\theta',0\}$.

Clearly $R'\{\theta'\}y$ and $R'\{\theta'\}x$ are two-sided ideals in $Spec_l R'\{\theta'\}$; and, since $R'\{\theta'\}x \cdot R'\{\theta'\}y = \{0\}$,

$$V_l(R'\{\theta'\}x) \bigcup V_l(R'\{\theta'\}y) = V_l(\{0\}) = Spec_l R'\{\theta'\}.$$

Futher, the quotient ring $R'\{\theta'\}/R'\{\theta'\}x$ is naturally isomorphic to the skew polynomial ring $R[y, \theta'^{-1}]$. Thus, we have canonical bijections (homeomorphisms):

$$V_l(R'\{\theta'\}x) \simeq Spec_l(R'\{\theta'\}/R'\{\theta'\}x) \simeq Spec_l R'[y, \theta'^{-1}].$$

Similarly,

$$V_l(R'\{\theta'\}y) \simeq Spec_l(R'\{\theta'\}/R'\{\theta'\}y) \simeq Spec_l R'[x, \theta'].$$

Since the quotient of the ring $R\{\theta'\}$ by the ideal $R'\{\theta'\}x + R'\{\theta'\}y$ is naturally isomorphic to the ring R', we have the canonical homeomorphisms:

$$V_l(R'\{\theta'\}x) \bigcap V_l(R'\{\theta'\}y) \simeq V_l(R'\{\theta'\}x + R'\{\theta'\}y) \simeq SpecR'.$$

Thus $Spec_l R'\{\theta'\}$ is the disjoint union of the closed subset $V_l(R'\{\theta'\}x + R'\{\theta'\}y)$, which is homeomorphic to $SpecR'$, and two open subsets:

$$V_l(R'\{\theta'\}x) - V_l(R'\{\theta'\}x + R'\{\theta'\}y),$$

and

$$V_l(R'\{\theta'\}y) - V_l(R'\{\theta'\}x + R'\{\theta'\}y),$$

which are homeomorphic respectively to $Spec_l R'[y, y^{-1}; \theta'^{-1}]$ and to $Spec_l R'[x, x^{-1}; \theta']$.

(b) Nondegenerate case: $\xi \notin p$. It follows from 3.2.5 that the subset of $Spec_l R\{\theta, \xi\}$ which consists of ideals of this type coincides with the preimage of $Spec_l K(R')[x, x^{-1}; \theta']$.

3.3. The restricted hyperbolic rings.

The only situation which is not covered by the analysis above is when $\theta(p) \neq p$, but $\theta^n(p) = p$ for some n, and $\theta^\nu(\xi) \notin p$ for all $\nu \in \mathbb{Z}$.

In this section, we consider an important special case - when $\theta^n = Id$ for some $n \geq 2$. The general case (in a much more general setting, for hyperbolic ring over noncommutative rings) is considered in Chapter IV.

3.3.1. Definition. Let θ be an automorphism of a commutative ring R such that $\theta^n = id$. And let ξ, $\mathfrak{u}$, and $\mathfrak{v}$ be elements in R having the properties:

$$\theta(\mathfrak{u}) = \mathfrak{u}, \quad \theta(\mathfrak{v}) = \mathfrak{v}, \quad \text{and} \quad \mathfrak{u}\mathfrak{v} = \prod_{1 \leq i \leq n} \theta^{i-1}(\xi).$$

The *restricted hyperbolic* ring, $R\{\theta, \xi \mid \mathfrak{u}, \mathfrak{v}, n\}$, is given by the relations:

$$xr = \theta(r)x, \quad ry = y\theta(r) \quad \text{for every} \quad r \in R; \quad xy = \xi,$$

$$x^n = u, \quad y^n = \mathfrak{v}. \tag{1}$$

3.3.2. Example. Let $R\{\theta,\xi\}$ be a hyperbolic ring, and $\theta^n = id$ for certain $n \geq 1$. Note that, thanks to the last equality, x^n and y^n commute with every element of R and between themselves. To check the latter property, notice that

$$x^n y^n = \prod_{1\leq i\leq n-1} \theta^i(\xi)$$

and

$$y^n x^n = \prod_{1\leq i\leq n} \theta^{-i}(\xi) = \theta^{-n}(\prod_{1\leq i\leq n-1} \theta^i(\xi)) = \prod_{1\leq i\leq n-1} \theta^i(\xi) = x^n y^n.$$

Thus, the ring $R\{\theta,\xi\}$ contains the polynomial subring $R[x^n, y^n]$ which we denote by $\mathfrak{R}$. Set $\mathfrak{u} = x^n$, $\mathfrak{v} = y^n$. Clearly the ring $R\{\theta,\xi\}$ is isomorphic to the restricted hyperbolic ring $\mathfrak{R}\{\theta,\xi \mid \mathfrak{u},\mathfrak{v},n\}$. ■

3.3.3. The left spectrum. Fix a restricted hyperbolic ring $R\{\theta,\xi \mid n\} = R\{\theta,\xi \mid \mathfrak{u},\mathfrak{v},n\}$. Since the elements $\mathfrak{u}$ and $\mathfrak{v}$ are central, we have the following decomposition of the left spectrum of the ring $R\{\theta,\xi \mid n\}$:

$$Spec_l R\{\theta,\xi \mid n\} = V_l(R\mathfrak{u} + R\mathfrak{v})\bigcup(V_l(\mathfrak{u})\bigcap U_l(\mathfrak{v}))\bigcup(V_l(\mathfrak{v})\bigcap U_l(\mathfrak{u}))\bigcup U_l(\mathfrak{u}\mathfrak{v}).$$

It is easy to see that

$$V_l(R\mathfrak{u} + R\mathfrak{v}) \simeq Spec(R/R\xi);$$

$$V_l(\mathfrak{u})\bigcap U_l(\mathfrak{v}) \simeq Spec_l(\mathfrak{v})^{-1}R[y,\theta^{-1} \mid \mathfrak{v},n] \simeq Spec_l R'[y,\vartheta^{-1} \mid \mathfrak{v},n],$$

where $R' = (\mathfrak{v})^{-1}R$, ϑ is the induced by θ automorphism of R', $\mathfrak{v}'$ is the image of $\mathfrak{v}$ in R', $R'[y,\vartheta^{-1} \mid \mathfrak{v}',n]$ is a restricted skew polynomial ring (cf. 3.3.1);

$$V_l(\mathfrak{v})\bigcap U_l(\mathfrak{u}) \simeq Spec_l(\mathfrak{u})^{-1}R[x,\theta \mid \mathfrak{u},n] \simeq Spec_l R"[x,\vartheta" \mid \mathfrak{u}",n],$$

where $R'' = (\mathfrak{u})^{-1}R, \vartheta''$ is the induced by θ automorphism of R'', $\mathfrak{u}''$ is the image of $\mathfrak{u}$ in R'';

$$U_l(\mathfrak{u}\mathfrak{v}) \simeq Spec_l(\mathfrak{u}\mathfrak{v})^{-1}R\{\theta,\xi \mid n\} \simeq Spec_l\mathfrak{R}\{\Theta,\xi\hat{} \mid \mathfrak{u}\hat{},\mathfrak{v}\hat{},n\},$$

where $\mathfrak{R} = (\mathfrak{u}\mathfrak{v})^{-1}R$, $\xi\hat{}$, $\mathfrak{v}\hat{}$, $\mathfrak{u}\hat{}$ are the images of ξ, $\mathfrak{v}$ and $\mathfrak{u}$ in $\mathfrak{R}$; Θ is the induced by θ automorphism of $\mathfrak{R}$. Note that, since the elements $\mathfrak{v}\hat{}$ and $\mathfrak{u}\hat{}$ are invertible, x, y and $\xi\hat{} = xy$ are invertible. In particular, $y = x^{-1}\xi\hat{}$. This implies that the ring $\mathfrak{R}\{\Theta,\xi\hat{} \mid n,\mathfrak{u}\hat{},\mathfrak{v}\hat{}\}$ is isomorphic to the restricted skew Laurent polynomial ring $\mathfrak{R}[x,x^{-1};\Theta \mid \mathfrak{u}\hat{},n]$. In particular, $U_l(\mathfrak{u}\mathfrak{v}) \simeq Spec_l\mathfrak{R}[x,x^{-1};\Theta \mid \mathfrak{u}\hat{},n]$.

4. Applications to basic examples.

In this Section, we apply the results of Section 3 to obtain the spectral picture of most popular classical and quantum algebras:

the quantum and classical enveloping algebras of the Lie algebra $sl(2)$;

the quantum coordinate algebra of $SL(2)$;
the first Weyl algebra;
the algebra of q-differential operators;
quantum plane.

For the last three algebras we show how to deduce from the description of the left spectrum a classification of irreducible representations. For the first Weyl algebra A_1, we show that any nonzero point of the left spectrum is closed; i.e. it is equivalent to a left maximal ideal. Thus we get almost for free (modulo results of Section 3) a description of irreducible representations of the first Weyl algebra which differs from the one given by R. Block [B1], [B2]. The difference is due to a different choice of parametrization: we use hyperbolic presentation of A_1 (i.e. the coordinate $\xi = xy$) which allows to simplify and enrich the description.

A generic algebra of q-differential operators (i.e. $q \neq 1, 0$) has quite a few of nonclosed points. As well as the quantum plane. Note by passing that the spectral picture of the algebra of q-differential operators is much closer (when $q \neq 1$) to that of the q-plane than to the spectral picture of A_1.

4.1. The quantum coordinate algebra of SL(2). Let $R\{\theta, \xi\}$ be the quantum coordinate algebra of $SL(2, k)$; i.e.

$$R = k[u, v], \quad \theta f(u, v) = f(qu, qv) \quad \text{for any } f \in R, \quad \xi = 1 + q^{-1}uv$$

(cf. Example 3.1.5).

Let p be a nonzero prime ideal of the ring $R = k[u, v]$; and let an ideal $\mathbf{p} \in Spec_l R\{\theta, \xi\}$ be such that $\mathbf{p} \bigcap R = p$.

(1) Principal series. Suppose that $x \in \mathbf{p}$. Then, the element

$$yx = \theta^{-1}(\xi) = 1 + q^{-3}uv$$

belongs to the ideal $p = \mathbf{p} \bigcap R$.

(1') If $R\{\theta, \xi\}/\mathbf{p}$ is of finite type over R, then

$$\vartheta^{m-1}(\xi) = 1 + q^{2m-3}uv \in p$$

for some $m \geq 1$. This and the inclusion $1 + q^{-3}uv \in p$ imply that

$$1 - q^{2m} \in p; \text{ i.e. } q^{2m} = 1.$$

In particular, if q is not a root of one, the principal series contains no representations of R-finite type.

(1") The representation $R\{\theta, \xi\}/\mathbf{p}$ of the principal series is not of finite type if and only if

$$\vartheta^n(\xi) = 1 + q^{2n-1}uv \notin p \quad \text{for any } n \geq 1.$$

Note that, since $\xi = 1 + q^{-1}uv \in p$, this implies that q is not a root of one.

(2) Discrete series. Let now $\mathbf{p} \in Spec_l R\{\theta, \xi\}$ be such that $\mathbf{p} \bigcap R = p$ and $\mathbf{p}$ does not contain any degrees of x or y. This means that

$$\xi = 1 + q^{-1}uv \notin p \quad \text{and} \quad \vartheta^n(\xi) - \xi = (q^{2n} - 1)q^{-1}uv \notin p$$

for any nonzero integer n. In other words, q is not a root of one, and the elements $u,\ v,\ 1 + q^{-1}uv$ do not belong to the ideal p.

4.1.1. Series of irreducible representations (the case of algebraically closed field). Suppose now that the field k is algebraically closed. Then every maximal ideal in the ring $R = k[u, v]$ is of the form $R(u - \lambda) + R(v - \eta)$, where λ, η are elements of the field k. It follows from 2.1 that

1) The ideal $p = R(u - \lambda) + R(v - \eta)$, $\lambda, \eta \in k$, defines a representation of principal series if and only if $\lambda \neq 0$ and $\eta = -q/\lambda$.

2) The ideal $p = R(u - \lambda) + R(v - \eta)$ defines the representation of the discrete series if and only if $\lambda \neq 0, \eta \neq 0$ and $\eta \neq -q/\lambda$.

4.2. The left spectrum and irreducible representations of $U_q(sl(2))$. Assume that

$$R = A[\xi],\ A \text{ is } \Theta\text{-stable, and } \Theta(\xi) = \xi + u \text{ for some } u \in A$$

This holds both for $U_q(sl(2))$ and $U(sl(2)))$. One can see that

$$\Theta^{n-1}(\xi) = \Theta^{-1}(\xi) + \sum_{0 \leq i \leq n-1} \Theta^{n-1-i}(u) \tag{1}$$

and

$$\Theta^{-n}(\xi) = \xi - \sum_{1 \leq i \leq n} \Theta^{i-n}(u) \tag{2}$$

for every $n \geq 1$.

Consider now the case of $U_q(sl(2))$; i.e. A is the algebra $k[z, z^{-1}]$ of Laurent polynomials in z; $\Theta f(z) = f(qz)$; $\quad u = (z - z^{-1})/(q - q^{-1})$. We assume, for simplicity, that q is not a root of one, and there is a square root of q in k; i.e. $q = \lambda^2$ for some $\lambda \in k$. The formulas (1), (2) in this case look as follows:

$$\Theta^n(\xi) = (\xi - u) + z^{-1}q(1 - q^{n+1})((q^2 - 1)(1 - q))^{-1}(z^2 - q^{-n}) \tag{3}$$

$$\Theta^{-n}(\xi) = \xi + z^{-1}q^2(1 - q^{-n})((q^2 - 1)(1 - q))^{-1}(z^2 - q^{-n+1}) \tag{4}$$

for any $n \geq 1$.

Fix a prime ideal p of the ring $R = A[\xi]$; and denote by p' the intersection $p \bigcap A$. Now we shall follow the pattern of Theorem 3.2.2.

Let $\xi - u \in p$. Then the ideal p is generated by the element $\xi - u$ and by the prime ideal $p' = p \bigcap A$ of the ring $A = k[z, z^{-1}]$:

$$p = A[\xi](\xi - u) + A[\xi]p'.$$

1) Suppose that $\xi \in p$. Then $u \in p'$ which implies that p' is generated either by $z-1$, or by $z+1$. Thus, we have two maximal ideals in $U_q(sl(2))$ of codimesion 1 which are generated by $x, y, z \pm 1$ respectively (cf. Theorem 3.2.2).

2) It follows from (3) that $\Theta^n(\xi) \in p$ if and only if

$$(z^2 - q^{-n}) = (z - \lambda^{-n})(z + \lambda^{-n}) \in p'$$

which means that either $p' = (z + \lambda^{-n})$, or $p' = (z - \lambda^{-n})$.

By Theorem 3.2.2, the left ideals of the ring $U_q(sl(2))$ generated by

$$\xi - u := xy - u, \; x, \; y^{n+1} \text{ and } (z + \lambda^{-n}) \text{ or } (z - \lambda^{-n})$$

are maximal; and one can see that the corresponding irreducible representations are $(n+1)$-dimesional.

Note that we can replace $\xi - u = \xi - u(z)$ in the list of generators above by $\xi - u(\pm\lambda^{-n})$.

Thus for each $n \geq 1$, there are exactly two n-dimensional representations. And this exhausts the list of finite dimensional representations of $U_q(sl(2))$.

3) Every irreducible polynomial $f(z)$ which is not equal to μz or to $\mu(z \pm \lambda^{-n})$ for any $\mu \in k - \{0\}$ and $n \geq 1$, defines two left maximal ideals:

the one generated by $\xi - u(z)$, x and $f(z)$;

and the one generated by ξ, y and $f(z)$.

The corresponding quotient modules are infinite dimensional (irreducible) representations of principal series.

Note that the left ideals

$$U_q(sl(2))(\xi - u(z)) + U_q(sl(2))x \quad \text{and} \quad U_q(sl(2))\xi + U_q(sl(2))y$$

are also in the left spectrum, but they are not maximal.

According to Theorem 3.2.3, every pair $\alpha, \gamma \in k$ such that $\gamma \neq 0$, and

$$\alpha \neq \gamma^{-1} q^2 (1 - q^{-n})((q^2 - 1)(1 - q))^{-1} (\gamma^2 - q^{-n+1})$$

for any integer n, defines a maximal left ideal

$$U_q(sl(2))(\xi - \alpha) + U_q(sl(2))(z - \gamma).$$

In the case when k is algebraically closed, these ideals exhaust the list of the left maximal ideals of $U_q(sl(2))$ which are generated by a prime ideal of the subring $A[\xi] = A[xy]$ (cf. Theorem 3.2.3). But, there are lots of non-closed points of the form $U_q(sl(2))p$, where $p \in SpecA[\xi]$.

The non-degenerate case remains (cf. 3.2.5): the ideals of the left spectrum which have zero intersection with the subalgebra $A[\xi] = k[z, z^{-1}, \xi]$. According to 3.2.5, this part of $Spec_l U_q(sl(2))$ is isomorphic to the left spectrum of the quantum plane without origin; i.e.

$$Spec_l U_q(sl(2)) = Spec_l k(z, \xi)_q[x, x^{-1}],$$

where the ring $k(z,\xi)_q[x,x^{-1}]$ of *q-Laurent polynomials* is defined by the relations:

$$xz = qzx, \quad x\xi = (\xi + u)x.$$

Now,

$$Spec_l k(z,\xi)_q[x,x^{-1}] = Spec_l k(z,\xi)_q[x] - \{0\}.$$

Each ideal of $Spec_l k(z,\xi)_q[x] - \{0\}$ is equivalent to a maximal left ideal (cf. 3.2.5); and any maximal left ideal is generated by an irreducible skew polynomial in x with coefficients in $k(z,\xi)$ which is not equal to μx for any $\mu \in k(z,\xi)^*$.

4.3. The classical case. Let now $R\{\Theta,\xi\} = U(sl(2))$; i.e. $R = k[z,\xi]$, and $\Theta f(z,\xi) = f(z+\alpha, \xi + z)$. Then

$$\Theta^{n-1}(\xi) = \Theta^{-1}(\xi) + \sum_{0\le i\le n-1} \Theta^{n-1-i}(z) = \Theta^{-1}(\xi) + nz + \frac{(n-1)n}{2}\alpha \tag{1}$$

and

$$\Theta^{-n}(\xi) = \xi - \sum_{1\le i\le n} \Theta^{i-n}(u) = \xi - nz - \frac{(n-1)n}{2}\alpha. \tag{2}$$

Here $n \ge 1$.

Repeating the same kind of analysis as for $U_q(sl(2))$, one can (with less effort) recover the spectral picture and the well known results of the representation theory of the Lie algebra $sl(2)$. Actually, this is the easiest known to me way to get the representation theory of $sl(2)$.

Assume that $char(k) = 0$.

Fix a prime ideal p of the ring $R = A[\xi] = k[z,\xi]$; and set $p' := p \bigcap A$. Again, we follow the pattern of Theorem 3.2.2.

(a) Let $\xi - z \in p$. Then the ideal p is generated by $\xi - z$ and by the prime ideal $p' = p\bigcap A$ of the ring $A = k[z]$: $\quad p = A[\xi](\xi - z) + A[\xi]p'$.

It follows from (1) that $\Theta^n(\xi) \in p$ if and only if $z + (n-1)\alpha/2 \in p'$.

By Theorem 3.2.2, the left ideal $p_{1,n}$ of the ring $U(sl(2))$ generated by

$$\xi - z := xy - z,\ x,\ y^n,\ \text{and } z + (n-1)\alpha/2$$

(or, what is the same, by $\xi + (n-1)\alpha/2,\ x,\ y^n$, and $z + (n-1)\alpha/2$) is maximal, and the corresponding irreducible representation $U(sl(2))/p_{1,n}$ is n -dimesional. Thus, we have for each $n \ge 1$ exactly one n-dimensional representation. And this exhausts the list of finite dimensional representations of $sl(2)$.

(b) Any irreducible polynomial $f(z)$ such that $f(n\alpha/2) \neq 0$ for all integers n, defines two left maximal ideals:

$p_{1,\infty}$ generated by $\xi - z, x$, and $f(z)$;

$p_{\infty,1}$ generated by $\xi,\ y$, and $f(z)$.

Here p is the maximal ideal in $k[z]$ generated by f.

The corresponding quotient modules are infinite dimensional (irreducible) representations of principal series. In particular, for any $\lambda \in k$ which is not equal to $n\alpha/2$ for any $n \in \mathbb{Z}$, we have the highest and the lowest weight representations (Verma modules) corresponding to the polynomial $z - \lambda$. In this case (which is general if k is algebraically closed),

$$p_{1,\infty} = (\xi - \lambda, x, z - \lambda) \text{ and } p_{\infty,1} = (\xi, y, z - \lambda).$$

(b1) Note that the left ideals

$$U(sl(2))(\xi - z) + U(sl(2))x \quad \text{and} \quad U(sl(2))\xi + U(sl(2))y$$

are also in the left spectrum, but they are not maximal.

(c) It follows from Proposition 3.2.3 that any maximal ideal p of the polynomial ring $k[z, \xi]$ such that $(\xi - nz - n(n-1)\alpha/2) \notin p$ for all $n \in \mathbb{Z}$, generates a left maximal ideal $p_{\infty,\infty} := U(sl(2))p$.

In the case the field k is algebraically closed, these are exactly ideals generated by $(\xi - \gamma)$, $(z - \lambda)$, where $\gamma \neq n\lambda + n(n-1)\alpha/2$ for any $n \in \mathbb{Z}$.

(c1) Every nonclosed point $p \in Speck[z, \xi]$ such that $(\xi - nz - n(n-1)\alpha/2) \notin p$ for any $n \in \mathbb{Z}$ generates a nonclosed point $Spec_l U(sl(2))$.

(d) Finally, there remains the non-degenerate case (cf. 3.2.5): the ideals of the left spectrum which have zero intersection with the subalgebra $A[\xi] = k[z, \xi]$. According to 3.2.5, this part of $Spec_l U(sl(2))$ is isomorphic to the left spectrum of the ring of skew Laurent polynomials $k(z, \xi)[x, x^{-1}; \theta]$, where θ acts on rational functions by $\theta f(z, \xi) = f(z + \alpha, \xi + z)$. Now,

$$Spec_l k(z, \xi)[x, x^{-1}; \theta] = Spec_l k(z, \xi)[x; \theta] - \{0\};$$

any ideal of $Spec_l k(z, \xi)[x; \theta] - \{0\}$ is equivalent to a maximal left ideal (cf. 3.2.5); and any maximal left ideal is generated by an irreducible skew polynomial $f(x)$ with coefficients in $k(z, \xi)$ such that $f(0) \neq 0$.

4.4. The left spectrum of the first Weyl algebra, quantum plane, the algebra of q-differential operators. Fix a field k, and consider the family of k-algebras $\mathbb{D}_{q,\hbar}$, where $(q, \hbar)$ is an arbitrary element of $k^* \times k$. The algebra $\mathbb{D}_{q,\hbar}$ is generated over k by elements x and y subject to the relation:

$$xy - qyx = \hbar \tag{1}$$

Thus, $\mathbb{D}_{q,0}$ is a quantum ('classical' if $q = 1$) plane; $\mathbb{D}_{1,1}(\simeq \mathbb{D}_{1,\hbar}$ if $\hbar \neq 0)$ is the first Weyl algebra which is isomorphic to the algebra $k[x, d/dx]$ of differential operators with polynomial coefficients. If $q \neq 1$, then the algebra $\mathbb{D}_{q,\hbar}$ is naturally realized as the algebra of *q-differential operators with polynomial coefficients* $k[y, d_q]$ which is the k-subalgebra of the algebra of endomorphisms of the k-module $k[z]$ of polynomials in z generated by the operator of multiplication $y : f(z) \longmapsto zf(z)$ and by the *q-derivative*

$$d_{q,\hbar} : f(z) \longmapsto \hbar(f(qz) - f(z))/(zq - z).$$

The algebra $\mathbb{D}_{q,\hbar}$ is isomorphic to the hyperbolic k-algebra $R\{\vartheta, \xi\} = k[\xi]\{\theta, \xi\}$, where the k-algebra automorphism θ is defined by $\theta(\xi) = q\xi + \hbar$.

Thus, if $q \neq 1$, then we have:

$$\theta^n(\xi) = q^n\xi + \hbar(1-q^n)/(1-q) = q^{n+1}\theta^{-1}(\xi) + \hbar(1-q^{n+1})/(1-q), \tag{2}$$

$$\theta^{-n}(\xi) = q^{-n}\xi + \hbar(1-q^{-n})/(1-q)$$

for every $n \geq 0$.

If $q = 1$, then

$$\theta^n(\xi) = \xi + n\hbar = \theta^{-1}(\xi) + (n+1)\hbar,$$

$$\theta^{-n}(\xi) = \xi - n\hbar$$

for every $n \geq 0$.

4.4.1. Quantum case. Consider first the case $q \neq 1$.

Fix a prime ideal p of the ring $R = k[\xi]$.

(a) Let $p = R(\xi - \hbar)$. If q is not a root of one, then

$$p_{1,\infty} := R\{\theta, \xi\}(\xi - \hbar) + R\{\theta, \xi\}x$$

is a left maximal ideal. One can see that the module $R\{\theta, \xi\}/p_{1,\infty}$ is isomorphic to the canonical representation of the ring $\mathbb{D}_{q,\hbar}$ as the ring of q-differential operators (see above).

If $q^m = 1$ for some $m \geq 2$ and $q^i \neq 1$ if $1 \leq i < m$, then

$$p_{1,m} := R\{\theta, \xi\}(\xi - \hbar) + R\{\theta, \xi\}x + R\{\theta, \xi\}y^m$$

is a left maximal ideal.

(b) Dually, if $p = R\xi$ and q is not a root of one, then

$$p_{\infty,1} = R\{\theta, \xi\}\xi + R\{\theta, \xi\}y$$

is a left maximal ideal.

(c) The maximal ideal Rf of the ring $R = k[\xi]$ is θ-stable if and only if $f(q\xi + \hbar) = \lambda f(\xi)$ for some $\lambda \in k^*$. One can see that the function $\xi - \hbar(1-q)^{-1}$ satisfies this property with $\lambda = q$; hence the ideal $R(\xi - \hbar(1-q)^{-1})$ is θ-stable. If q is not a root of one, this is the only possibility.

Suppose that this is not the case, and $f(\xi)$ is a polynomial in ξ such that $\theta f(\xi) = \lambda f(\xi)$ for some $\lambda \in k^*$. We can represent f in the form

$$f(\xi) = \sum_{0 \leq i \leq m} a_i(\xi - \gamma)^i,$$

where $\gamma := \hbar(1-q)^{-1}$, and $a_0 \neq 0$. Then

$$\lambda f(\xi) = \theta f(\xi) = f(\theta(\xi)) = \sum_{0 \leq i \leq m} a_i q^i (\xi - \gamma)^i.$$

Since $a_0 \neq 0$, $\lambda = 1$. This implies that $q^i = 1$ for every i such that $a_i \neq 0$.

The quotient ring $R/R(\xi - \gamma)$ is isomorphic to k; and the corresponding quotient hyperbolic ring is defined by the equations:

$$xy = \hbar(1-q)^{-1} = yx;$$

i.e. the quotient hyperbolic ring is a hyperbola over k. So, its spectrum coincides with $Speck[x, x^{-1}]$.

The same argument shows that, if q is not a root of one, then every θ^n-stable maximal ideal in $k[\xi]$ is generated by the element $\xi - \hbar(1-q)^{-1}$; in particular, it is θ-stable.

(d) If q is not a root of one, then every irreducible polynomial $f(\xi)$ which is not equal to $\mu(\xi - \hbar(1-q)^{-1})$ or to $\lambda(\xi - \hbar(1-q^{-n})/(1-q))$ for some integer n generates a left maximal ideal, $R\{\theta, \xi\}f$.

(e) There is a natural embedding of $Max_l k(\xi)[x, x^{-1}; \vartheta]$ into $Spec_l R\{\theta, \xi\}$, where ϑ is the induced by θ automorphism of the field $k(\xi)$: $\vartheta(\xi) = q\xi + \hbar$. Every irreducible element g of the ring $k(\xi)[x; \vartheta]$ such that $g(0) \neq 0$ generates a left maximal ideal in $k(\xi)[x, x^{-1}; \vartheta]$.

In particular, every polynomial $x - f$, where $f = f(\xi) \in k(\xi)^*$, generates a left maximal ideal in the ring $k(\xi)[x, x^{-1}; \vartheta]$.

4.4.2. The classical case. Consider now the case $q = 1$, $\hbar \neq 0$; i.e. $\mathbb{D}_{q,\hbar}$ is the first Weyl algebra. Then

$$\theta^n(\xi) = \xi + n\hbar = \theta^{-1}(\xi) + (n+1)\hbar,$$

$$\theta^{-n}(\xi) = \xi - n\hbar$$

for every $n \geq 0$.

Fix a prime ideal p of the ring $R = k[\xi]$.

(a) Let $p = R(\xi - \hbar)$. If $char(k) = 0$, then

$$p_{1,\infty} := R\{\theta, \xi\}(\xi - \hbar) + R\{\theta, \xi\}x$$

is a left maximal ideal. One can see that the module $R\{\theta, \xi\}/p_{1,\infty}$ is isomrorphic to the canonical representation of the ring $\mathbb{D}_{q,\hbar}$ as the ring of q-differential operators (see above).

If $char(k) = \mathfrak{p} > 0$, then

$$p_{1,\mathfrak{p}} := R\{\theta, \xi\}(\xi - \hbar) + R\{\theta, \xi\}x + R\{\theta, \xi\}y^{\mathfrak{p}}$$

is a left maximal ideal.

(b) Dually, if $p = R\xi$, and $char(k) = 0$, then

$$p_{\infty,1} = R\{\theta, \xi\}\xi + R\{\theta, \xi\}y$$

is a left maximal ideal.

(c) The maximal ideal $k[\xi]f$ of the ring $k[\xi]$ is θ^n-stable if and only if $f(\xi+n) = \lambda f(\xi)$ for some $\lambda \in k^*$. Clearly $\lambda = 1$; i.e. $f(\xi)$ itself is θ^n-stable. Now, the equality $f(\xi + n) = f(\xi)$ implies that $n = l \cdot char(k)$ for some integer l.

Consider the whole picture in the case when $chark = 0$. Then there is no θ^n-stable non-constant polynomials for any $n \neq 0$. According to Theorem 3.2.2, every irreducible polynomial $f(\xi)$ which is not equal to $\mu(\xi - n)$ for some $n \in \mathbb{Z}$ and $\mu \in k^*$ generates a left maximal ideal $R\{\theta, \xi\}f$. This fact implies a theorem by Dixmier [D2].

There is a natural imbedding of $Max_l k(\xi)[x, x^{-1}; \vartheta]$ into $Spec_l R\{\theta, \xi\}$, where ϑ is the induced by θ automorphism of the field $k(\xi) : \vartheta(\xi) = \xi + 1$. Every irreducible element g of the ring $k(\xi)[x; \vartheta]$ such that $g(0) \neq 0$ generates a left maximal ideal in $k(\xi)[x, x^{-1}; \vartheta]$. In particular, every polynomial $x - f$, where $f = f(\xi) \in k(\xi)^*$, generates a left maximal ideal in the ring $k(\xi)[x, x^{-1}; \vartheta]$.

4.4.3. Proposition. *Any nonzero element of* $Spec_l \mathbb{D}_{1,\hbar}$, $\hbar \neq 0$, *is equivalent to a maximal left ideal.*

Proof. *a)* Fix an abelian category $\mathcal{A}$. For any $\mathbf{P} \in \mathbf{Spec}\mathcal{A}$, define the *height* $\mathfrak{ht}(\mathbf{P})$ of $\mathbf{P}$ as the supremum of nonnegative integers n such that there exists a chain $\mathbf{P} \supset \mathbf{P}_1 \supset \mathbf{P}_2 \supset \ldots \supset \mathbf{P}_n$ of distinct elements of $\mathbf{Spec}\mathcal{A}$. Here we take the canonical realization of $\mathbf{Spec}\mathcal{A}$ (cf. 0.3.1). Now we define the *spectral dimension*, $\dim_s \mathcal{A}$, of $\mathcal{A}$ as the supremum of $\mathfrak{ht}(\mathbf{P})$ while $\mathbf{P}$ runs through $\mathbf{Spec}\mathcal{A}$.

4.4.3.1. Proposition. *Suppose that a ring R has a finite Krull dimension. Then* $\dim_s \mathcal{A} \leq KdimR$.

Proof. The assertion follows immediately from Corollary 6.4.2 in Chapter VI. ∎

4.4.3.2. Corollary. *Let R be a prime ring of Krull dimension* 1. *Then any nonzero ideal of $Spec_l R$ is either equivalent to zero, or to a left maximal ideal.*

b) The Krull dimension of the first Weyl algebra is 1. Therefore, by Corollary 4.4.3.2, all those nonzero ideals of $Spec_l \mathbb{D}_{1,\hbar}$ which have zero intersection with the subring $k[\xi] = k[xy]$ are equivalent to left maximal ideals. ∎

Thus we have recovered (in a slightly different terms) the Richard Block's classification of irreducible representations of the first Weyl algebra [B1].

4.4.4. The quantum plane. This is the algebra $\mathbb{D}_{q,0}$ denoted usually by $k_q[x, y]$. The action of θ is very simple: $\theta(\xi) = q\xi$. Clearly the ideals 0 and (ξ) are θ-stable. If q is generic (i.e. not a root of one), or the field k is algebraically closed, then these two ideals are the only θ-stable primes in $R = k[\xi]$. If q is generic, then, for any n, 0 and (ξ) are the only θ^n-stable primes in R.

Consider the generic case: q is not a root of one. We have the following picture:

a) The quotient ring $R/R\xi$ is isomorphic to k; and the corresponding hyperbolic ring is a commutative k-algebra with generators x, y subject to the relations:

$$xy = yx = 0.$$

Its spectrum is naturally homeomorphic to $Speck[x] \coprod_{Speck} Speck[y]$.

b) The corresponding to 0 part of the left spectrum is $Spec_l k(\xi)\{\vartheta, \xi\}$, where ϑ is the extension of θ. And $k(\xi)\{\vartheta, \xi\} \simeq k(\xi)[x; \vartheta]$. So that

$$Spec_l k(\xi)\{\vartheta, \xi\} - (0) \simeq Spec_l k(\xi)[x; \vartheta] - (0) \approx Max_l k(\xi)[x; \vartheta];$$

and any left maximal ideal of $k(\xi)[x;\vartheta]$ is a principal ideal generated by an irreducible element of $k(\xi)[x;\vartheta]$.

c) The remaining part of the left spectrum consists of all ideals of the form $k_q[x,y]f$, where f runs through the set of all irreducible polynomials in ξ such that $f(0)\neq 0$.

4.4.5. The quantum torus. By definition, the quantum torus $\mathbf{T}_q$ is the k-module of q-Laurent polynomials $k_q[x,x^{-1},y,y^{-1}]$ with the multiplication defined by the same relation $xy=qyx$. Clearly the algebra $\mathbf{T}_q$ is isomorphic to the localization of the quantum plane at the multiplicative system (ξ) generated by $\xi=xy$. Therefore $Spec_l\mathbf{T}_q$ is the complement to the closed subset $V_l(\xi)=Spec_l k_q[x,y]/(\xi)$ of the left spectrum of the quantum plane. That is $Spec_l\mathbf{T}_q$ consists of the pieces *b)* and *c)* of $Spec_l k_q[x,y]$ (cf. 4.4.4).

Note that all points of $Spec_l\mathbf{T}_q$, except of the generic point 0, are closed. This can be showed by a 'Krull dimension' argument similar to that of Proposition 4.4.3, or can be proved directly as follows.

Let $r=\sum x^i a_i$ be an element of $\mathbf{T}_q$ which is irreducible as an element of $k(\xi)[x;\vartheta]$. So that the generated by r left ideal in $k(\xi)[x;\vartheta]$ is maximal. Denote by *(r)* the intersection of this left ideal with the algebra $\mathbf{T}_q$.

a) If the generated by the coefficients $\{a_i\}$ of r ideal in $k[\xi]$ coincides with $k[\xi]$, then the left ideal (r) is maximal.

In fact, if (r) is not maximal, it contains properly in a maximal left ideal μ which has a nontrivial intersection with $k[\xi]$. This implies that $\mu=\mathbf{T}_q f$ for some irreducible polynomial f. In particular, all coefficients a_i should belong to the ideal $k[\xi]f$ which contradicts to the hypothesis.

b) Consider now the general case: the coefficients $\{a_i\}$ of r generate a proper ideal $k[\xi]g$ for some polynomial g. Since $(r)\bigcap k[\xi]=0$, and (r) belongs to the left spectrum, the ideal $((r):g)$ is equivalent to (r). Note that $((r):g)=(r_1)$, where $r_1=r/g=\sum x^i a_i/g$, and the coefficients $\{a_i/g\}$ generate $k[\xi]$. Therefore the ideal (r_1) is maximal. ■

4.4.6. About closed points of a quantum plane. It follows from Proposition 3.2.3 that all the ideals of the series *c)* are maximal. There is also an obvious set of closed points; namely the preimage of the set of closed points of $V_l(\xi)=Spec_l k_q[x,y]/(\xi)$.

Examples of nonclosed points:

– The generic point which is the zero ideal.

– The left ideals generated respectively by x and y. They are preimages of the corresponding ideals in the commutative quotient ring $k_q[x,y]/(\xi)$ (cf. *a)* above); in particular, both are two-sided. The set of specializations on (x) (resp. (y)) is the preimage of $Speck[y]$ (resp. $Speck[x]$).

It is natural to ask how to distinguish closed points among those elements of the left spectrum which have zero intersection with $k[\xi]$.

4.4.6.1. Lemma. *(a) A left ideal μ in $k_q[x,y]=k[\xi]\{\theta,\xi\}$ is not contained in any of maximal ideals containing $\xi=xy$ if and only if μ has an element of the form $1+\xi\varphi$ for some $\varphi\in k_q[x,y]$.*

(b) Let $f(\xi)$ be any irreducible polynomial not proportional to ξ. Then a left ideal $\mu\in k_q[x,y]$ is not contained in the left ideal $k_q[x,y]f(\xi)$ iff μ has an element of the form $1+\varphi f(\xi)$ for some $\varphi\in k_q[x,y]$.

Proof. *(a)* We represent elements of the ring $k_q[x,y]$ as functions

$$f(x,y;\xi) = \sum_{m\geq 0} x^m a_m + \sum_{n\geq 1} y^n b_n \tag{2}$$

where $\{a_m, b_n\}$ are polynomials in ξ (cf. Lemma 3.1.6). Consider the set of functions $\mu' := \{f(x,y;0) \mid f \in \mu\}$. Clearly μ' is an ideal in the ring of factors $\mathcal{R} := k_q[x,y]/(\xi)$. And the ideal μ is not contained in any maximal ideal of the form (1) iff μ' contains the identity element of the ring $\mathcal{R}$. This means exactly that μ contains an element of the form $1 + \xi\varphi$ for some $\varphi \in k_q[x,y]$.

The inverse implication is evident.

(b) If $f(\xi)$ is an irreducible polynomial not proportional to ξ, then $k_q[x,y]f$ is a left maximal ideal. Therefore a left ideal μ is not contained in $k_q[x,y]f$ iff $1 \in \mu + k_q[x,y]f$. ∎

4.4.6.2. Corollary. *A left ideal $\mu \in Spec_l k_q[x,y]$ such that $\mu \bigcap k[\xi] = 0$ is a closed point (i.e. is equivalent to a maximal left ideal) if and only if, for any irreducible polynomial $f(\xi)$, μ contains elements of the form $1 + \varphi f(\xi)$ for some $\varphi \in k_q[x,y]$.*

Corollary 4.4.6.2 provides a recipe of creating closed points: just take any element $r \in k_q[x,y]$ of the form $1+\varphi\xi$, $\varphi \in k_q[x,y]$, which is an irreducible element in $k(\xi)[x, x^{-1}; \vartheta]$ (we replace y by $x^{-1}\xi$) and such that the ideal in $k[\xi]$ generated by coefficients of the decomposition $r = r(x,y;\xi) = \sum_{m\geq 0} x^m a_m + \sum_{n\geq 1} y^n b_n$ coincides with $k[\xi]$. Then the left ideal $(r) := k(\xi)\{\vartheta, \xi\} r \bigcap k_q[x,y]$ represents a closed point of the left spectrum; i.e. there exists an element f in $k[\xi]$ such that the left ideal $((r) : f)$ is maximal.

4.4.7. Closed and nonclosed points of $\mathbb{D}_{q,\hbar}$. Consider the algebra $\mathbb{D}_{q,\hbar}$ in the generic case: $q, \hbar \in k^*$, $q \neq 1$. Assume that $char(k) = 0$.

The description of closed and nonclosed points of the 'diagonalizable' part of the left spectrum (i.e. those $\mathbf{p} \in Spec_l \mathbb{D}_{q,\hbar}$ for which $\mathbf{p} \bigcap k[\xi] \neq 0$) is immediate: the only nonclosed points are the generic point 0 and the two-sided ideal generated by $\xi - \gamma$, where $\gamma = \hbar/(1-q)$ (cf. 4.4.1).

As in the case of quantum plane, we have a couple of canonical families of nonclosed points of the spectrum having only one specialization. These are principal left ideals generated by irreducible polynomials f in x or in y such that $f(0) \neq 0$.

Finally, we have the same elementary criteria for a non-diagonalizable element of the left spectrum of $\mathbb{D}_{q,\hbar}$ to represent a closed point:

4.4.7.1. Lemma. *A left ideal $\mathbf{p} \in Spec_l k_q[x,y]$ such that $\mathbf{p} \bigcap k[\xi] = 0$ is a closed point (i.e. is equivalent to a maximal left ideal) if and only if it contains elements of the form $1 + \varphi(\xi - \gamma)$ for some $\varphi \in k_q[x,y]$ and the coefficients of elements of $\mathbf{p}$ generate the ring $k[\xi]$.*

Here $\gamma = \hbar/(1-q)$.

Proof is analogous to that of Lemma 4.4.6.1.

4.4.8. An observation. When $q \in k - \{0,1\}$, the spectral picture of $\mathbb{D}_{q,\hbar}$ for $h \neq 0$ differs from the spectral picture of the quantum plane, $\mathbb{D}_{q,0}$, only in the commutative part: the hyperbola $xy = yx = \hbar/(1-q)$ splits into two axes.

But, the difference between $Spec_l \mathbb{D}_{q,\hbar}$ when $q \neq 1$ and the left spectrum of the first Weyl algebra $\mathbb{D}_{1,\hbar}$ is considerable.

Complementary facts and examples.

Section 4 covers only a part of the given in Introduction list of quantized (and classical) algebras. And the list itself is far from being complete.

The purpose of this appendix is to give to a reader a better view of a host of concrete 'small' algebras acquiring an importance in (not only) mathematical physics. Needless to say that the remaining examples from Introduction are included.

Also, the 'Complementary facts' (together with Section 4) might be regarded as a sort of a handbook on examples of hyperbolic rings which are of interest nowadays. In a couple of cases, we sketch (using the general results of Section 3 and some specific properties of the algebras in question) spectral pictures. But, mostly, we just give formulas needed to make the application of Theorem 3.2.2 and Porposition 3.2.3 straightforward leaving details to the reader.

In Sections C1, C2, C3, we study the left spectrum of special classes of hyperbolic rings. These classes are:

rings *of* $M(2)$*-type* (algebras $M_q(2)$, $SL_q(2)$, $GL_q(2)$ are of $M(2)$-type);

rings *of Heisenberg type* (the quantum Heisenberg [KS], [Ma], and Weyl [H] algebras are principal examples);

and, finally, rings *of* $U(sl(2)$*-type* the particular cases of which are the enveloping algebra $U(sl(2))$, the quantum enveloping algebra $U_q(sl(2))$, and the introduced in [S] "algebras similar to $U(sl(2))$".

Section C4 is concerned with the left spectrum of those 3-dimensional algebras (in the sense of [BS]) which happen to be hyperbolic, but do not belong to any of the listed above classes. Among them, the dispin algebra (– the universal enveloping algebra of the Lie superalgebra $osp(1,2)$) and the introduced by Woronowicz [W1] *twisted* $U(sl(2))$.

C1. Hyperbolic rings of $\mathbf{M(2)}$-type. Fix a hyperbolic ring $R\{\theta,\xi\}$ over a commutative noetherian ring R.

C1.1. Lemma. *1) The following properties of a θ-invariant element γ of the ring $R\{\theta,\xi\}$ are equivalent:*

(a) $\theta(\xi)+\gamma\theta^{-1}(\xi)=(\gamma+1)\xi$.

(b) $\gamma\theta^{-1}(\xi)-\xi$ is a central element in the ring $R\{\theta,\xi\}$.

2) If $\xi-\theta^{-1}(\xi)$ is not a zero divisor, then the element γ in (a) and (b) is uniquely defined.

Proof. 1) (a)⇒(b). We shall show that the element $\gamma\theta^{-1}(\xi)-\xi$ is central in $R\{\theta,\xi\}$. Note that, since $\gamma\theta^{-1}(\xi)-\xi\in R$, it is central if and only if it is θ-stable. The last property follows immediately from (1):

$$\theta(\gamma\theta^{-1}(\xi)-\xi)=\gamma\xi-\theta(\xi)=\gamma\xi+\gamma\theta^{-1}(\xi)-(\gamma+1)\xi=\gamma\theta^{-1}(\xi)-\xi.$$

(b) ⇒ (a). Conversely, the fact that $\lambda\theta^{-1}(\xi)-\xi$ and λ are θ-stable is expressed by the equality:

$$\lambda\xi-\theta(\xi)=\lambda\theta^{-1}(\xi)-\xi,$$

which is equivalent to (1) with γ replaced by λ.

2) Note that the equality *(a)* is equivalent to the equality

$$\theta(\xi - \theta^{-1}(\xi)) = \theta(\xi) - \xi = \gamma(\xi - \theta^{-1}(\xi)). \tag{1}$$

This means that if the element $u := \xi - \theta^{-1}(\xi)$ is not a zero divisor, then the element γ is uniquely defined. In particular, if the ring R is a domain, then either the element ξ itself is θ-stable, or the element γ (if any) is uniquely defined. ■

Thus, if R is a domain and $\theta(\xi) \neq \xi$, then the central element

$$\sigma(\theta, \xi) = \gamma\theta^{-1}(\xi) - \xi \tag{2}$$

is uniquely defined by θ and ξ.

C1.2. A special case: the ring $A\langle\vartheta, u\rangle$. Let $R\{\theta, \xi\}$ be the corresponding to the ring $A\langle\vartheta, u\rangle$ hypebolic ring (cf. 3.2.1): i.e. $R := A[\xi]$, $\theta\,|_A = \vartheta$, $\theta(\xi) = \xi + \vartheta(u)$, $\theta^{-1}(\xi) = \xi - u$.

Then $\xi - \theta^{-1}(\xi) = u$; and the condition *(a)* in Lemma C1.1 is equivalent to the equality

$$\vartheta(u) = \gamma u \tag{1}$$

(cf. the part 2) of the proof of Lemma C1.1).

Note that it follows from (1) that γ is an element of the ring A.

C1.3. Example: the coordinate ring of quantum 2 × 2 matrices. The *coordinate algebra* $\mathcal{A}(M_q(2))$ is the k-algebra with generators x, y, w, v and with the relations

$$\begin{gathered} qwx = xw, \quad qvx = vx, \quad qyw = wy, \\ qyv = vy, \quad wv = vw, \\ xy - yx = (q - q^{-1})wv \end{gathered} \tag{1}$$

These relations describe the algebra $A\langle\vartheta, u\rangle$, where

$$A = k[w, v], \quad \vartheta f(w, v) = f(qw, qv), \quad u = (q - q^{-1})wv.$$

The corresponding hyperbolic algebra, $R\{\theta, \xi\} = R\{x, y; \theta, \xi\}$, is given by

$$R = k[w, v, \xi], \quad \text{and} \quad \theta f(w, v, \xi) = f(qw, qv, \xi + (q^3 - q)wv)$$

for any polynomial $f(w, v, \xi)$.

Note that

$$\theta^{-1} f(w, v, \xi) = f(q^{-1}w, q^{-1}v, \xi - (q - q^{-1})wv).$$

Clearly $\vartheta(u) = q^2 u$; i.e. the ring $R\{\theta, \xi\}$ satisfies the conditions of Lemma C1.1. ■

Giving priority to this example, we shall say about a hyperbolic rings with the property *(a)* (or *(b)*) from Lemma C1.1 that they are of $M(2)$-type.

C1.4. The left spectrum of a hyperbolic ring of M(2)-type. Now fix a hyperbolic ring, $R\{\theta,\xi\}$, of $M(2)$-type. And denote by u the element $\xi-\theta^{-1}(\xi)$.

The equalities

$$\theta^{-1}(\xi)=\xi-u$$

and

$$\theta(\xi)=\xi+\gamma u=\theta^{-1}(\xi)+(1+\gamma)u$$

imply that

$$\theta^{i}(\xi)=\xi+(\sum_{1\le j\le i}\gamma^{j})u=\theta^{-1}(\xi)+(\sum_{0\le j\le i}\gamma^{j})u \tag{1}$$

and

$$\theta^{-i}(\xi)=\xi-(\sum_{0\le j\le i}\gamma^{j})u=\theta^{-1}(\xi)-(\sum_{1\le j\le i}\gamma^{j})u \tag{2}$$

for every positive integer i.

First consider special situations.

C1.4.1. The degenerate case: $u=0$. This means that $\xi=\theta(\xi)$, or, equivalently, ξ is a central element in $R\{\theta,\xi\}$. In particular, $R\{\theta,\xi\}\xi$ is a θ-stable two-sided ideal. Thus, we have the partition of the left spectrum of the ring $R\{\theta,\xi\}$:

$$Spec_l R\{\theta,\xi\}=V_l(\xi)\bigcup U_l(\xi),$$

and

$$V_l(\xi)\simeq Spec_l(R\{\theta,\xi\}/R\{\theta,\xi\}\xi), U_l(\xi)\simeq Spec_l((\xi)^{-1}R\{\theta,\xi\}).$$

The quotient ring $R\{\theta,\xi\}/R\{\theta,\xi\}\xi$ is isomorphic to the ring $(R/R\xi)\{\theta',0\}$, where θ' is the ring automorphism on $R/R\xi$ induced by θ; i.e. it is defined by the relations:

$$xr=\theta'(r)x,\quad ry=y\theta'(r)\quad\text{for every } r\in R/R\xi;$$
$$xy=0=yx.$$

Thus, $Spec_l(R/R\xi)\{\theta',0\}$ is naturally homeomorphic (with respect to any natural topology we might consider) to the push-forward

$$Spec_l(R/R\xi)[x,\theta']\coprod_{\mathfrak{r},\mathfrak{n}}Spec_l(R/R\xi)[y,\theta^{-1}].$$

It remains the open part of the decomposition (3) - the left spectrum of the localized ring $(\xi)^{-1}R\{\theta,\xi\}$.

Note that, thanks to the θ-invariance of ξ, the ring $(\xi)^{-1}R\{\theta,\xi\}$ is isomorphic to the ring $((\xi)^{-1}R)\{\theta\hat{},\xi\hat{}\}$, where $\theta\hat{}$ is the induced by θ automorphism of the ring $(\xi)^{-1}R$, $\xi\hat{}$ is the image of ξ in $(\xi)^{-1}R$. The equations $xy=\xi\hat{}=yx$ imply that the elements x, y are invertible, and $y=x^{-1}\xi\hat{}$. This means that the hyperbolic ring $(\xi)^{-1}R\{\theta\hat{},\xi\hat{}\}$ is isomorphic to the skew polynomial ring $(\xi)^{-1}R[x,\theta\hat{}]$.

Thus, in the degenerate case, $\xi = \theta(\xi)$, the description of the left spectrum of hyperbolic rings is reduced to the description of the left spectrum of skew polynomial rings, which is already known (cf. Section 1).

C1.4.2. The nondegenerate case. Suppose now that *the element* u *is invertible*. Then γ is also invertible, since θ is an automorphism. Now the θ-stable elespectrum:

$$Spec_l R\{\theta,\xi\} = V_l(1-\gamma)\bigcup U_l(1-\gamma),$$

and

$$V_l(1-\gamma) \simeq Spec_l(R\{\theta,\xi\}/R\{\theta,\xi\}(1-\gamma)),$$

$$U_l(1-\gamma) \simeq Spec_l(1-\gamma^{-1}R\{\theta,\xi\}).$$

Consider each of the spaces in the right side of the last two expressions.

C1.4.2.1. The left spectrum of $R\{\theta,\xi\}/R\{\theta,\xi\}(1-\gamma)$. Note that, again, since $1-\gamma$ is θ-stable,

$$R\{\theta,\xi\}/R\{\theta,\xi\}(1-\gamma) \simeq R''\{\theta'',\xi''\},$$

where $R'' = R/R(1-\gamma)$, θ'' is the induced by θ automorphism of R'', ξ'' the image of ξ in R''.

Clearly the hyperbolic ring $R''\{\theta'',\xi''\}$ is of $M(2)$-type, but, since the image of γ in R'' is 1, the element

$$u'' = \xi'' - \theta''^{-1}(\xi'')$$

is θ''-stable (cf. C1.3). Therefore, the formulas (1), (2) (cf. the beginning of the section C1.4) acquire, in this case, a particularly simple form:

$$\theta''^n(\xi) = \xi + nu'' = \theta''^{-1}(\xi) + (n+1)u'' \tag{5}$$

and

$$\theta''^{-n}(\xi) = \xi - (n+1)u'' = \theta''^{-1}(\xi) - nu'' \tag{6}$$

for every positive integer n.

We leave to the reader the application of Theorem 3.2.2 and Proposition 3.2.3 to this case.

C1.4.2.2. The open part. Since the element $1-\gamma$ is θ-stable,

$$(1-\gamma)^{-1}R\{\theta,\xi\} \simeq \mathfrak{R}\{\Theta,\xi'\},$$

where $\mathfrak{R} := (1-\gamma)^{-1}R$ – the localization of the ring R at $1-\gamma$, Θ the automorphism induced by θ,ξ' the image of ξ.

Set $u' := \xi' - \Theta(\xi')$; and let γ' denote the image of γ in R. It follows from the formulas (1), (2) (cf. the beginning of C1.4) that

$$\Theta^i(\xi) = \xi + \gamma'(1-\gamma')^{-1}(1-\gamma'^i)u' = \Theta^{-1}(\xi) + (1-\gamma')^{-1}(1-\gamma'^{i+1})u' \tag{7}$$

and

$$\Theta^{-i}(\xi) = \xi - (1-\gamma')^{-1}(1-\gamma'^{i+1})u' = \Theta^{-1}(\xi) - \gamma'(1-\gamma')^{-1}(1-\gamma'^{i+1})u' \qquad (8)$$

C1.4.3. General case. Let now $R\{\theta,\xi\}$ be a generic ring of $M(2)$-type, $u := \xi - \theta^{-1}(\xi)$, $\theta(u) = \gamma u$. Then, thanks to the last property, $R\{\theta,\xi\}u$ is a two-sided ideal, and the quotient ring, $R\{\theta,\xi\}/R\{\theta,\xi\}u$ is isomrphic to the hyperbolic ring $R'\{\theta',\xi'\}$, where $R' = R/Ru, \theta'$ is the induced by θ automorphism, ξ' the image of ξ in R'.

The equality $\theta(u) = \gamma u$ implies that the multiplicative subset $(u) := \{u^n \mid n \geq 0\}$ is an Ore set. The localization of $R\{\theta,\xi\}$ at (u) is isomorphic to the hyperbolic ring $R\hat{}\{\theta\hat{},\xi\hat{}\}$, where $R\hat{} = (u)^{-1}R, \theta\hat{}$ is the induced by θ automorphism of $R\hat{}, \xi\hat{}$ is the image of ξ. Clearly the ring $R\hat{}\{\theta\hat{},\xi\hat{}\}$ is also of $M(2)$-type, and the image $u\hat{}$ of the element u is invertible.

Thus, we have the decomposition

$$Spec_l R\{\theta,\xi\} = V_l(u)\bigcup U_l(u) \simeq Spec_l R'\{\theta',\xi'\}\bigcup Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\},$$

in which the hyperbolic ring $R'\{\theta',\xi'\}$ is *degenerate*, i.e. $\xi' = \theta'(\xi')$ (cf. C1.4.1), and $R\hat{}\{\theta\hat{},\xi\hat{}\}$ is nondegenerate (cf. C1.4.2).

C1.5. The left spectrum of the ring $A(M_q(2))$. Recall that the coordinate algebra, $A(M_q(2))$, of quantum 2×2 matrices is the ring $A\langle\vartheta,u\rangle$, where

$$A = k[w,v], \quad \vartheta f(w,v) = f(qw,qv), \quad u = (q-q^{-1})wv.$$

(cf. Example C1.2). The corresponding hyperbolic ring, $R\{\theta,\xi\}$ is given by:

$$R = k[w,v,\xi], \quad \theta f(w,v,\xi) = f(qw,qv,\xi + (q^3-q)wv) \qquad (1)$$

for any polynomial $f(w,v,\xi)$.

Clearly

$$\gamma = q^2 : \theta(u) = \vartheta(u) = q^2 u.$$

According to C1.4.3,

$$Spec_l R\{\theta,\xi\} = V_l(u)\bigcup U_l(u),$$

and

$$V_l(u) \simeq Spec_l R'\{\theta',\xi'\}, U_l(u) \simeq Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\},$$

where

$$R' = k[w,v,\xi]/\{vw\} \simeq (k[v]\prod k[w])[\xi'], \qquad (2)$$

and

$$\theta' f(v,w,\xi') = f(qv,qw,\xi') \quad \text{for every } f(v,w,\xi') \in R';$$

$$R\hat{} \simeq k[w,w^{-1},v,v^{-1},\xi\hat{}], \qquad (3)$$

and

$$\theta\hat{}f(w,v,\xi) = f(qw,qv,\xi\hat{} + (q^3 - q)wv)$$

(cf. (1)). According to C1.4.1,

$$Spec_l R'\{\theta',\xi'\} = (Spec_l R"[x,\theta"] \coprod_{SpecR"} Spec_l R"[y,\theta"^{-1}]) \bigcup Spec_l(\xi')^{-1}R'\{\theta',\xi'\},$$

where

$$R" = R'^{/} R'\xi' \simeq k[v] \coprod k[w], \quad \text{and} \quad \theta"(f(v,w) = f(qv,qw) \quad \text{for any} \quad f(v,w) \in R";$$

and

$$(\xi')^{-1}R'\{\theta',\xi'\} \simeq ((\xi')^{-1}R')[x,\theta'] \simeq (k[v] \coprod k[w])[\xi',\xi'^{-1}][x,\theta'].$$

Therefore

$$Spec_l R"[x,\theta"] \coprod_{SpecR"} Spec_l R"[y,\theta"^{-1}] \simeq$$

$$Spec_l k_q[v,x] \coprod_{Speck[v]} Spec_l k_{\mathfrak{v}}[v,y^{-1}] \coprod_{Speck} Spec_l k_q[v,x] \coprod_{Speck[v]} Spec_l k_{\mathfrak{v}}[w,y^{-1}], \tag{4}$$

where $\mathfrak{v} = q^{-1}$, and

$$Spec_l(\xi')^{-1}R'\{\theta',\xi'\} \simeq$$

$$Spec_l k_q[x,v,v^{-1}] \coprod_{Speck[v]} Spec_l k_q[x,w,w^{-1}]. \tag{5}$$

Now consider the nondegenerate part of the left spectrum, i.e. the open subset $Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\}$, where $R\hat{} \simeq k[w,w^{-1},v,v^{-1},\xi\hat{}]$, and $\theta\hat{}f(w,v,\xi\hat{}) = f(qw,qv,\xi\hat{}+(q^3-q)wv)$ (cf. (3)).

(a) If $\gamma = q^2 = 1$, then, for every positive integer n,

$$\theta\hat{}^n(\xi\hat{}) = \xi\hat{} + nu\hat{} = \theta\hat{}^{-1}(\xi\hat{}) + (n+1)u\hat{}, \tag{6}$$

and

$$\theta\hat{}^{-n}(\xi\hat{}) = \xi\hat{} - (n+1)u\hat{} = \theta\hat{}^{-1}(\xi\hat{}) - nu\hat{}, \tag{7}$$

where $u\hat{} = (q-q^{-1})vw$. The general theorems of Section 3 provide the following assertion.

C1.5.1. Proposition. *Let* $char(k) = 0$. *Then*

1) To every prime ideal p *in* $k[w,w^{-1},v,v^{-1}]$, *there correspond two non-equivalent left ideals from* $Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\}$:

$$p_{\infty,1} = R\hat{}\{\theta\hat{},\xi\hat{}\}p + R\hat{}\{\theta\hat{},\xi\hat{}\}\xi\hat{} + R\hat{}\{\theta\hat{},\xi\hat{}\}y$$

and

$$p_{1,\infty} = R\hat{}\{\theta\hat{},\xi\hat{}\}p + R\hat{}\{\theta\hat{},\xi\hat{}\}(\xi\hat{} - u\hat{}) + R\hat{}\{\theta\hat{},\xi\hat{}\}x.$$

Every $\mathbf{p} \in Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$ *such that* $\xi\hat{} - nu\hat{} \in \mathbf{p}$ *for some* $n \geq 1$ *(resp.* $n \leq 0$*) is equivalent to* $p_{1,\infty}$ *(resp. to* $p_{\infty,1}$*) for some ideal* p *from* $Speck[w, w^{-1}, v, v^{-1}]$.

2) Let an ideal $\mathfrak{p} \in Speck[w, w^{-1}, v, v^{-1}, \xi\hat{}]$ *be such that, for every nonzero integer* n*, there is an* $f(v, w, \xi\hat{})$ *in* $\mathfrak{p}$ *such that* $f(v, w, \xi\hat{} - nu\hat{}) \notin \mathrm{p}$*, and* $\xi\hat{} - iu\hat{} \notin \mathfrak{p}$ *for all integers* i.

Then every left ideal $\mathbf{p}$ *in* $R\hat{}\{\theta\hat{}, \xi\hat{}\}$ *such that*

$$\mathbf{p} \bigcap R\hat{}\{\theta\hat{}, \xi\hat{}\} = \mathfrak{p}$$

coincides with $\mathfrak{p}_{\infty,\infty} := R\hat{}\{\theta\hat{}, \xi\hat{}\}\mathfrak{p}$*, and* $\mathfrak{p}_{\infty,\infty} \in Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$.

(b) Suppose now that $\gamma := q^2 \neq 1$. Then

$$\theta\hat{}^{\,i}(\xi\hat{}) = \xi\hat{} + \gamma(1-\gamma)^{-1}(1-\gamma^i)u\hat{}$$

and

$$\theta\hat{}^{\,-i}(\xi\hat{}) = \xi\hat{} - (1-\gamma)^{-1}(1-\gamma^{i+1})u\hat{}$$

for every positive integer i. Here the specialization of general facts looks as follows.

C1.5.2. Proposition. *Let* q *(hence* $\gamma = q^2$*) be not a root of one. Then*

1*) To every* $p \in Speck[w, w^{-1}, v, v^{-1}]$*, correspond two non-equivalent left ideals from* $Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$*:*

$$p_{\infty,1} = R\hat{}\{\theta\hat{}, \xi\hat{}\}p + R\hat{}\{\theta\hat{}, \xi\hat{}\}\xi\hat{} + R\hat{}\{\theta\hat{}, \xi\hat{}\}y$$

and

$$p_{1,\infty} = R\hat{}\{\theta\hat{}, \xi\hat{}\}p + R\hat{}\{\theta\hat{}, \xi\hat{}\}(\xi\hat{} - \gamma u\hat{}) + R\hat{}\{\theta\hat{}, \xi\hat{}\}x.$$

Every $\mathbf{p} \in Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$ *such that*

$$\xi\hat{} + \gamma(1-\gamma)^{-1}(1-\gamma^i)u\hat{} \in \mathbf{p}$$

$$(resp.\ \xi\hat{} - (1-\gamma)^{-1}(1-\gamma^i)u\hat{} \in \mathbf{p})$$

for some $i \geq 1$ *(resp.* $i \geq 0$*) is equivalent to* $p_{1,\infty}$ *(resp. to* $p_{\infty,1}$*) for certain* $p \in Speck[w, w^{-1}, v, v^{-1}]$.

2) Let a prime ideal $\mathfrak{p}$ *in* $k[w, w^{-1}, v, v^{-1}, \xi\hat{}]$ *does not contain neither* $\xi\hat{} + \gamma(1-\gamma)^{-1}(1-\gamma^i)u\hat{}$*, nor* $\xi\hat{} - (1-\gamma)^{-1}(1-\gamma^i)u\hat{}$ *for any integer* $i \geq 0$*, but, for every* $i \geq 1$ *and* $n \geq 0$*, there exist an* $f(v, w, \xi\hat{})$ *and an* $g(v, w, \xi\hat{})$ *in* $\mathfrak{p}$ *such that both*

$$f(v, w, \xi\hat{} + \gamma(1-\gamma)^{-1}(1-\gamma^i)u\hat{}) \quad and \quad g(v, w, \xi\hat{} - (1-\gamma)^{-1}(1-\gamma^i)u\hat{})$$

do not belong to $\mathfrak{p}$.

Then every left ideal $\mathbf{p}$ *in* $R\hat{}\{\theta\hat{}, \xi\hat{}\}$ *such that*

$$\mathbf{p} \bigcap R\hat{}\{\theta\hat{}, \xi\hat{}\} = \mathfrak{p}$$

coincides with $\mathfrak{p}_{\infty,\infty} := R\hat{}\{\theta\hat{},\xi\hat{}\}\mathfrak{p}$, *and* $\mathfrak{p}_{\infty,\infty} \in Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\}$.

C1.6. Coordinate algebras of SL_q and GL_q. The *quantum determinant*,

$$\xi - quv = xy - quv = yx - q^{-1}uv,$$

is a θ -stable element in $R\{\theta, \xi\}$:

$$\theta(\xi - quv) = \xi + (q^3 - q)uv - q^3 uv = \xi - quv$$

which means exactly that $\xi - quv$ is a central element.

Note that if q is not a root of one, then the center, $\mathcal{C}(\mathcal{A}(M_q(2)))$, of the algebra $\mathcal{A}(M_q(2))$ is generated by the *'quantum determinant'*.

The coordinate ring $\mathcal{A}(SL_q(2))$ is the quotient of the coordinate ring $\mathcal{A}(M_q(2))$ of quantum 2×2 matrices by the ideal generated by $\xi - quv - 1$; i.e. the algebra $\mathcal{A}(SL_q(2))$ is obtained from $\mathcal{A}(M_q(2))$ by adding the relation:

$$xy - quv = 1$$

(cf. 4.1).

The coordinate ring $\mathcal{A}(GL_q(2))$ is the localization of the coordinate ring $\mathcal{A}(M_q(2))$ of quantum 2×2 matrices at the powers of the determinant.

It is more convenient to describe these two algebras in terms of the corresponding hyperbolic rings.

In fact, if $R\{\theta, \xi\}$ is the hypebolic ring of $M_q(2)$, and $\det_q := \xi - quv$ is the quantum determinant, then the quotient of $R\{\theta, \xi\}$ by the ideal generated by $(\det_q - 1) = \xi - quv - 1$ is naturally isomorphic to a hyperbolic ring $A\{\vartheta, \zeta\}$, where

$$A = k[u, v], \quad \vartheta f(u, v) = f(qu, qv); \quad \zeta = 1 + quv$$

Similarly, the localization of $R\{\theta, \xi\}$ at $(\det_q)$ is just the hyperbolic ring $R'\{\theta', \xi'\}$, where R' is the localization of the (commutative) ring R at $(\det_q)$, θ' is the unique extension of the automorphism θ onto R', ξ' is the image of ξ under the canonical morphism $R \to R'$.

C2. Quantum Heisenberg algebra. Given nonzero elements, q and ρ, of a field k, denote by $\mathcal{H}(q, \rho)$ the k-algebra generated by indeterminates x, y, z subject to the following relations:

$$xz = q^{-1}zx; \quad yz = qzy. \tag{1}$$

$$xy - \rho yx = z. \tag{2}$$

The algebra $\mathcal{H}(q, \rho)$ is a two-parameter deformation of the Heisenberg algebra. Clearly $\mathcal{H}(q, \rho)$ is one of the simplest examples of the rings $A\langle\vartheta, \rho, u\rangle$ (cf. 3.1.10):

$$A = k[z], \vartheta f(z) = f(q^{-1}z) \quad \text{for any polynomial } f(z), \quad u = z.$$

The corresponding hyperbolic algebra is $R\{\theta, \xi\}$, where

$$R = k[z, \xi], \quad \theta f(z, \xi) = f(q^{-1}z, \rho\xi + q^{-1}z) \quad \text{for any } f \in k[z, \xi]$$

Note that $\theta^{-1}f(z,\xi) = f(qz, \rho^{-1}(\xi - z))$. In particular, we have:

$$\theta(\xi) + \theta^{-1}(\xi) = (\rho + \rho^{-1})\xi + (q^{-1} - \rho^{-1})z. \tag{3}$$

The equality (3) suggests that a special choice of parameters, namely $\rho = q$, might have some advantages. And this is really the case, as the following Proposition shows:

C2.1. Proposition. *Let θ be an automorphism of a commutative ring R. Suppose that an element ξ of R satisfies the condition*

$$\theta(\xi) + \theta^{-1}(\xi) = (\rho + \rho^{-1})\xi \tag{4}$$

for some invertible element ρ such that $\theta(\rho) = \rho$.

Then $(\xi - \rho^{-1}\theta^{-1}(\xi))(\xi - \rho\theta^{-1}(\xi))$ is a central element in the hyperbolic algebra $R\{\theta, \xi\} = R\{x, y; \theta, \xi\}$.

Proof. We have:

$$\theta(\xi - \rho\theta^{-1}(\xi)) = \theta(\xi) - \rho\xi = \rho^{-1}\xi - \theta^{-1}(\xi) = \rho^{-1}(\xi - \rho\theta^{-1}(\xi)). \tag{5}$$

Since ρ and ρ^{-1} enter symmetrically into the equation (4), it follows from (5) that

$$\theta(\xi - \rho^{-1}\theta^{-1}(\xi)) = \rho(\xi - \rho^{-1}\theta^{-1}(\xi)). \tag{6}$$

Therefore

$$(\xi - \rho^{-1}\theta^{-1}(\xi))(\xi - \rho\theta^{-1}(\xi))$$

is a θ-stable (hence central) element. ■

The algebra $\mathcal{H}(q,q)$ was introduced in [KS] as a q-analog of the Heisenberg algebra (in [KS], it is denoted by H_q). The prime spectrum of this algebra is studied in [Ma].

C2.2. 'Heisenberg type' Hyperbolic algebras. Now, instead of direct investigation of the rings $H_q = \mathcal{H}(q,q)$, we shall look at properties of a more general class of Hyperbolic rings which arises from Proposition C2.1.

That class consists of Hyperbolic rings $R\{\theta, \xi\}$ such that

$$\theta(\xi) + \theta^{-1}(\xi) = (\rho + \rho^{-1})\xi \tag{1}$$

for some θ-stable invertible element ρ. We shall refer to the hyperbolic rings with property (1) as *hyperbolic rings of Heisenberg type.*

Note that the class hyperbolic rings of Heisenberg type is stable under the adjunction (cf. 3.1.9); i.e. the adjoint to $R\{\theta, \xi\}$ ring, $R\{\theta^{-1}, \theta^{-2}(\xi)\}$, also satisfies the condition (1):

$$(\theta^{-1} + \theta)(\theta^{-2}(\xi)) = \theta^{-2}(\theta^{-1} + \theta)(\xi) = \theta^{-2}((\rho + \rho^{-1})\xi) = (\rho + \rho^{-1})\theta^{-2}(\xi).$$

C2.3. A special case: rings $A\langle\vartheta, \rho, u\rangle$ of Heisenberg type. Let ϑ be an outomorphism of a commutative ring A, ρ an invertible element of A, and $A\langle\vartheta, \rho, u\rangle$ a ring defined by the relations:

$$xa = \vartheta(a)x, ay = y\vartheta(a) \quad \text{for every } a \in A,$$

$$xy - \rho yx = u \quad \text{for certain } u \in A.$$

Finally, let $R\{\theta, \xi\}$ be the associated with $A\langle\vartheta, \rho, u\rangle$ hyperbolic ring:

$$R = A[\xi], \quad \theta(\xi) = \vartheta(\rho)\xi + \vartheta(u), \quad \theta\,|_A = \vartheta.$$

(cf. Section 3.1.10).

Since $\theta^{-1}(\xi) = \rho^{-1}(\xi - u)$, we have:

$$\theta(\xi) + \theta^{-1}(\xi) = (\vartheta(\rho) + \rho^{-1})\xi + (\vartheta(u) - \rho^{-1}u). \tag{1}$$

The equality (1) implies that

$$\theta(\xi) + \theta^{-1}(\xi) = (\rho + \rho^{-1})\xi \tag{2}$$

if and only if

$$\vartheta(\rho) = \rho \quad \text{and} \quad \vartheta(u) = \rho^{-1}u. \tag{3}$$

C2.3.1. Example. Let $A = k[z]$, $\vartheta f(z) = f(q^{-1}z)$, as in C1.3; but let $u = hz^n$ for some nonnegative integer n and an $h \in k^*$. Then (3) holds if and only if $\rho = q^n$. In particular, $\rho = 1$ if $u = h \in k$.

Similarly, we can take, instead of $A = k[z]$, the algebra of Laurent polynomials, $A = k[z, z^{-1}]$, with the same sort of action, $\vartheta f(z) = f(q^{-1}z)$, and with $u = hz^n$ for some $h \in k$ and an integer n. Then (3) holds if and only if $\rho = q^n$.

Note that if q is not a root of one, then the only solutions of the system (3) are $u = hz^n$, $\rho = q^n$, $n \geq 0$.

In fact, the equality $\vartheta(\rho) = \rho$ means exactly that $\rho \in k$. Since q is not a root of one, the equality $\vartheta(u) = \rho^{-1}u$ implies that $u = hz^n$ for some $h \in k$ and some integer n. ■

C2.4. A canonical central element. Let $R\{\theta, \xi\}$ be a hyperbolic ring of Heisenberg type; i.e.

$$(\theta + \theta^{-1})(\xi) = (\rho + \rho^{-1})\xi \quad \text{and} \quad \theta(\rho) = \rho. \tag{1}$$

for an invertible element ρ such that $\theta(\rho) = \rho$.

By Proposition C2.1,

$$\mathfrak{c}(\rho) = (\xi - \rho^{-1}\theta^{-1}(\xi))(\xi - \rho\theta^{-1}(\xi)) \tag{2}$$

is a central element in the ring $R\{\theta, \xi\}$.

C2.4.1. The case of rings $A\langle\vartheta, \rho, u\rangle$. If $R\{\theta, \xi\}$ is the hyperbolic ring associated with the ring $A\langle\vartheta, \rho, u\rangle$ (cf. C2.3), then

$$\mathfrak{c}(\rho) = (xy - \rho^{-1}yx)u. \tag{1}$$

In particular, if $A = k[z]$ (or if $A = k[z, z^{-1}]$), then, necessarily, $u = hz^n$ and $\rho = q^n$ (cf. Example C2.3.1), and

$$\mathfrak{c}(\rho) = h(xy - q^{-n}yx)z^n. \tag{2}$$

It follows from the equalities $\theta(u) = \rho^{-1}u$ and $\vartheta(\rho) = \rho$ that

$$\mathfrak{c}(\rho) = (xy - \rho^{-1}yx)u = xyu - \rho^{-1}y\theta(u)x = xyu - \rho^{-1}y\rho^{-1}ux = xyu - \rho^{-2}yux$$

I.e.

$$\mathfrak{c}(\rho) = x(yu) - \rho^{-1}(yu)x. \tag{3}$$

The equality (3) shows that if the element u is invertible, then the ring $A\langle\vartheta, \rho, u\rangle$ is isomorphic to the ring $A\langle\vartheta, \rho^{-1}\mathfrak{c}(\rho)\rangle$.

C2.5. The hyperbolic rings and the rings $A\langle\vartheta, \rho, u\rangle$ of Heisenberg type. Let, for a short while, $R\{\theta, \xi\}$ be a generic hyperbolic ring, and let ρ be an invertible element in R. Denote by u the element $\xi - \rho\theta^{-1}(\xi)$, and consider the ring $R\langle\theta, \rho, u\rangle$. Let $R[t]\{\Theta, t\}$ be the associated with $R\langle\theta, \rho, u\rangle$ hyperbolic algebra (cf. C2.3): $\Theta\mid_R = \theta, \quad \Theta(t) = \theta(\rho)t + \theta(u)$.

Clearly the map $\varphi_\rho : R[t] \longrightarrow R$ which is identical on R and sends t into ξ, is a ring epimorphism such that $\varphi_\rho \circ \Theta = \theta \circ \varphi_\rho$. Therefore φ_ρ defines the canonical ring epimorphism

$$\psi_\rho : R[t]\{\Theta, t\} \longrightarrow R\{\theta, \xi\}.$$

Now suppose that the hyperbolic ring $R\{\theta, \xi\}$ is of Heisenberg type, and let ρ be an element such that

$$(\theta + \theta^{-1})(\xi) = (\rho + \rho^{-1})\xi \quad \text{,and} \quad \theta(\rho) = \rho. \tag{1}$$

It follows from (1) that

$$\theta u := \theta(\xi - \rho\theta^{-1}(\xi)) = \theta(\xi) - \rho(\xi) = \rho^{-1}\xi - \theta^{-1}(\xi) = \rho^{-1}u$$

Since $\theta(\rho) = \rho$ and $\theta(u) = \rho^{-1}u$, the ring $R\langle\theta, \rho, u\}$, or, what is the same, the associated hyperbolic ring $R[t]\{\Theta, t\}$, is of Heisenberg type. Note also that

$$\Theta(t) = \rho t + \rho^{-1}u. \tag{2}$$

C2.6. The left spectrum of a hyperbolic ring of Heisenberg type. Fix a hyperbolic ring $R\{\xi, \theta\}$ of Heisenberg type; i.e.

$$\theta(\xi) + \theta^{-1}(\xi) = (\rho + \rho^{-1})\xi.$$

And set, as above, $u := \xi - \rho\theta^{-1}(\xi)$.

C2.6.1. Lemma. *For any nonnegative integer n,*

$$\theta^n(\xi) = \rho^{n+1}\theta^{-1}(\xi) + \rho^{-n}\Big(\sum_{0\le i\le n} \rho^{2i}\Big)u, \tag{1}$$

and

$$\theta^{-n-1}(\xi) = \rho^{-n-1}\xi - \rho^{-n-1}\Big(\sum_{0\le i\le n} \rho^{2i}\Big)u. \tag{2}$$

Proof. When $n = 0$, the formula (1) is just the definition of u: $\xi = \rho\theta^{-1}(\xi) + u$. If (1) holds for some n, then, thanks to the equalities

$$\theta(\rho) = \rho \quad \text{and} \quad \theta(u) = \rho^{-1}u,$$

it holds for $n + 1$:

$$\theta^{n+1}(\xi) = \theta(\rho^{n+1}\theta^{-1}(\xi) + \rho^{-n}(\sum_{0\leq i\leq n} \rho^{2i})u) =$$

$$\rho^{n+1}\xi + \rho^{-n}(\sum_{0\leq i\leq n} \rho^{2i})\theta(u) =$$

$$\rho^{n+2}\theta^{-1}(\xi) + \rho^{n+1}u + \rho^{-n}(\sum_{0\leq i\leq n} \rho^{2i})\rho^{-1}u =$$

$$\rho^{n+2}\theta^{-1}(\xi) + \rho^{-n-1}(\sum_{0\leq i\leq n+1} \rho^{2i})u.$$

The formula (2) follows from the formula (1) for the conjugate ring, $R\{\theta^{-1}, \theta^{-2}(\xi)\}$ with ρ^{-1} instead of ρ and, as a consequence, $-u$ instead of u. This is an easy way to write it. But, once the formula is written, it is easier to prove it by induction. The details are left to a reader. ■

C2.6.2. Decompositions. Thanks to the property $\theta(u) = \rho^{-1}u$, the left ideal $R\{\theta, \xi\}u$ is, actually, two-sided. Thus, we have the decomposition

$$Spec_l R\{\theta, \xi\} = V_l(u) \bigcup U_l(u),$$

and

$$V_l(u) \simeq Spec_l R'\{\theta', \xi'\}, U_l(u) \simeq Spec_l R"\{\theta", \xi"\},$$

where $R' = R/Ru, \theta'$ is the induced by θ automorphism of R', ξ' is the image of ξ;
$R" = (u)^{-1}R, \theta"$ is the induced by θ automorphism of $R"$, $\xi"$ is the image of ξ.

C2.6.2.1. $Spec_l R'\{\theta', \xi'\}$**.** Since $\theta'(\xi') = \rho'\xi'$, where ρ' is the image of ρ in R', the left ideal $R'\{\theta', \xi'\}\xi'$ is two-sided. Therefore

$$Spec_l R'\{\theta', \xi'\} = V_l(\xi') \bigcup U_l(\xi'),$$

and

$$V_l(\xi') \simeq Spec_l(R'/R'\xi')\{\theta', 0\}; U_l(\xi') \simeq Spec_l(\xi')^{-1}R'\{\theta', \xi'\}.$$

The left spectrum of $(R'/R'\xi')\{\theta', 0\}$ is homeomorphic to

$$Spec_l(R'/R'\xi')[x, \theta'] \coprod_{SpecR'/R'\xi'} Spec_l(R'/R'\xi')[y, \theta'^{-1}].$$

And

$$(\xi')^{-1}R'\{\theta',\xi'\} \simeq (\xi')^{-1}R'[x,x^{-1};\theta'].$$

(cf. C1.4.1).

C2.6.2.2. *$Spec_l R"\{\theta",\xi"\}$*. The image, $u"$, of the element u is invertible. Therefore, following the scenario of C1.4.2, we consider the decomposition with respect to the central element $1-\rho"^2$, where $\rho"$ is the image of the element ρ in $R"$:

$$Spec_l R"\{\theta",\xi"\} = V_l(1-\rho"^2)\bigcup U_l(1-\rho"^2),$$

and

$$V_l(1-\rho"^2) \simeq Spec_l R\{\theta,\xi\},$$

where $R = R"/R"(1-\rho"^2)$;

$$U_l(1-\rho"^2) \simeq Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\},$$

where $R\hat{} = (1-\rho"^2)^{-1}R"$.

Now it remains only the open set $U_l(1-\rho"^2) \simeq Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\}$.

Denote by $u\hat{}$ and $\rho\hat{}$ the images in $R\hat{}\{\theta\hat{},\xi\hat{}\}$ of u and ρ respectively. Since the element $1-\rho\hat{}^2$ is invertible, we can rewrite the formulas (1), (2) from Lemma C2.6.1 as

$$\theta\hat{}^n(\xi) = \rho\hat{}^{n+1}\theta\hat{}^{-1}(\xi) + \rho\hat{}^{-n}(1-\rho\hat{}^{2(n+1)})(1-\rho\hat{}^2)^{-1}u\hat{} \tag{1}$$

and

$$\theta\hat{}^{-n-1}(\xi) = \rho\hat{}^{-n-1}\xi - \rho\hat{}^{-n-1}(1-\rho\hat{}^{2(n+1)})(1-\rho\hat{}^2)^{-1}u\hat{}. \tag{2}$$

The formulas (1), (2) provide a specialization of Theorem 3.2.2 and Proposition 3.2.3 which we formulate here for readers' convenience.

Set $\mathfrak{ch}(p,\lambda)=0$ if $1-\lambda^i \notin p$ for all $i \geq 1$; otherwise it is equal to the minimal positive integer i such that $1-\lambda^i \in p$.

C2.6.2.2.1. Proposition. *(a) Let $p \in Spec R\hat{}$.*

1) If $\theta\hat{}^{-1}(\xi) \in p$, and $\mathfrak{ch}(p,\rho\hat{}^2) = \mathfrak{r}$, then the left ideal

$$p_{1,\mathfrak{r}+1} = R\{\theta\hat{},\xi\hat{}\}p + R\{\theta\hat{},\xi\hat{}\}x + R\{\theta\hat{},\xi\hat{}\}y^{\mathfrak{r}+1}$$

belongs to $Spec_l R\{\theta,\xi\}$.

2) If $\theta\hat{}^{-1}(\xi) \in p$, and $\mathfrak{ch}(p,\rho\hat{}^2) = 0$, then the left ideal

$$p_{1,\infty} = R\{\theta\hat{},\xi\hat{}\}p + R\{\theta\hat{},\xi\hat{}\}x$$

is in $Spec_l R\hat{}\{\theta\hat{},\xi\hat{}\}$.

3) If $\xi \in p$, and ch $(p,\rho\hat{}^2) = 0$, *then the left ideal*

$$p_{\infty,1} = R\{\theta\hat{},\xi\hat{}\}p + R\{\theta\hat{},\xi\hat{}\}y$$

is in $Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$.

4) Suppose that

$$\rho\hat{}^{n+1}\theta\hat{}^{-1}(\xi) + \rho\hat{}^{-n}(1-\rho\hat{}^{2(n+1)})(1-\rho\hat{}^{2})^{-1}u\hat{} \notin p$$

and

$$\theta\hat{}^{-n-1}(\xi) = \rho\hat{}^{-n-1}\xi - \rho\hat{}^{-n-1}(1-\rho\hat{}^{2(n+1)})(1-\rho\hat{}^{2})^{-1}u\hat{} \notin p$$

for every nonnegative integer n; and let p be not $\theta\hat{}^{m}$*-stable for any nonzero integer m. Then the left ideal* $p_{\infty,\infty} := R\{\theta, \xi\}p$ *belongs to* $Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$.

(b) All the listed above left ideals are not equivalent one to another.

(c) Every left ideal $\mathbf{p}$ *from* $Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$ *which contains*

$$\rho\hat{}^{n+1}\theta\hat{}^{-1}(\xi) + \rho\hat{}^{-n}(1-\rho\hat{}^{2(n+1)})(1-\rho\hat{}^{2})^{-1}u\hat{}$$

for some integer $n \geq 0$ *is equivalent either to* $p_{1,\mathfrak{r}+1}$*, or to* $p_{1,\infty}$ *for certain* $p \in Spec R\hat{}$.

(d) Every left ideal $\mathbf{p}$ *from* $Spec_l R\hat{}\{\theta\hat{}, \xi\hat{}\}$ *which contains*

$$\theta\hat{}^{-n-1}(\xi) = \rho\hat{}^{-n-1}\xi - \rho\hat{}^{-n-1}(1-\rho\hat{}^{2(n+1)})(1-\rho\hat{}^{2})^{-1}u\hat{} \notin p$$

for some integer $n \geq 0$ *is equivalent either to* $p_{1,\mathfrak{r}+1}$*, or* $p_{\infty,1}$ *for certain* $p \in Spec R\hat{}$.

C2.6.3. A Version of Engel's theorem. Recall that, given a ring B and its subring A, a B-module is called A-finite if it is finitely generated as an A-module.

One of the consequences of the obtained above description of the left spectrum of Heisenberg type hyperbolic rings is the following fact:

C2.6.3.1. Proposition. *Let* $R\{\theta, \xi\}$ *be a hyperbolic ring of Heisenberg type with the weight* ρ*. Suppose that the subring of* θ*-invarinant elements of the ring R is a field, and* ρ *is not a root of one.*

Then the following properties of a left ideal $\mathbf{p}$ *from* $Spec_l R\{\theta, \xi\}$ *are equivalent:*

(a) the quotient module $R\{\theta, \xi\}/\mathbf{p}$ *is R-finite;*

(b) $\mathbf{p} = p + R\{\theta, \xi\}x + R\{\theta, \xi\}y$ *for some* $p \in Spec R$.

In particular, $\mathbf{p}$ *is two-sided, and* $R\{\theta, x\}/\mathbf{p} \simeq R/(R\xi + R\theta^{-1}(\xi))$.

C2.6.3.2. Corollary. *Let* $R\{\theta, \xi\}$ *be as in Proposition C2.6.3.*1.

1*) If* $\alpha := R\xi + R\theta^{-1}(\xi)$ *is a proper ideal in R, then a simple left* $R\{\theta, \xi\}$*-module is R-finite if and only if it is isomorphic, as an R-module, to the quotient module* R/μ *with zero action of x and y, where* μ *is a maximal ideal in R which contains* α.

2) If $R\xi + R\theta^{-1}(\xi)$ *coincides with R, then there are no nonzero R-finite* $R\{\theta, \xi\}$*-modules.*

C2.8. A version of skew Weyl algebras. T. Hayashi [H] has defined a quantum version of the first Weyl algebra as the ring A_q generated over a field k by x, y, z subject to the relations

$$xz = q^{-1}zx; \quad yz = qzy; \tag{5}$$

$$xy - \rho yx = z; \tag{6}$$

and

$$\mathfrak{c}(q) = (xy - q^{-1}yx)z = 1. \tag{7}$$

By analogy, consider the ring, $R\{\theta, \rho, \xi\}$, obtained from $R\{\theta, \xi\}$ by adding the relation:

$$\mathfrak{c}(\rho) = (\xi - \rho^{-1}\theta^{-1}(\xi))(\xi - \rho\theta^{-1}(\xi)) = 1. \tag{8}$$

The ring $R\{\theta, \rho, \xi\}$ shall be called *the Weyl ring associated with* $R\{\theta, \xi\}$ or, shortly, the Weyl ring, when it does not create ambiguity.

C2.8.1. The ring $WA\langle\vartheta, \rho, u\rangle$. Consider the special case, – when $R\{\theta, \xi\}$ is the hyperbolic ring associated with the ring $A\langle\vartheta, \rho, u\rangle$ of Heisenberg type (cf. C2.4). Then the associated Weyl ring, $R\{\theta, \rho, \xi\}$, can be described in terms A, ϑ, u and ρ (using the canonical isomorphism $A\langle\vartheta, \rho, u\rangle \longrightarrow R\{\theta, \xi\}$) as the ring generated by x, y and A satisfying the relations:

$$xa = \vartheta(a)x, \quad ay = y\vartheta(a) \quad \text{for every } a \in A, \tag{1}$$

$$xy - \rho yx = u \quad \textit{for certain } u \in A. \tag{2}$$

$$\mathfrak{c}(\rho) = (xy - \rho^{-1}yx)u = 1. \tag{3}$$

The ring defined by the relations (1), (2), (3) will be denoted by $WA\langle\vartheta.\rho, u\rangle$.

Clearly the ring $WA\langle\vartheta, 1, u\rangle$ coincides with $WA\langle\vartheta, 1, 1\rangle$; i.e. this ring is given by the relations:

$$xa = \vartheta(a)x, \quad ay = y\vartheta(a) \quad \text{for every } a \in A, \tag{1}$$

$$xy - yx = 1. \tag{4}$$

In particular, the (conventional) first Weyl algebra is a subalgebra of $WA\langle\vartheta, 1, 1\rangle$.

C3. Rings of $\mathbf{U(sl(2))}$-type. A ring $A\langle\vartheta, u\rangle$ will be said to be *of $U(sl(2))$-type* if there exists an element $\mathfrak{v}$ in A such that

$$\mathfrak{v} - \vartheta^{-1}(\mathfrak{v}) = u. \tag{1}$$

Clearly the solution of (1) is determined uniquely up to a ϑ-invariant summand. We shall call any solution of (1) a *weight* of the ring $A\langle\vartheta, u\rangle$.

C3.2. Example. Let $A\langle\vartheta, u\rangle = U_q(sl(2))$; i.e. $A = k[z, z^{-1}], \vartheta f(z) = f(qz)$ for some $q \in k - \{0, \pm 1\}$, $u = (z - z^{-1})/(q - q^{-1})$.

Then

$$\mathfrak{v} = (qz + z^{-1})(q - 1)^{-1}(q - q^{-1})^{-1} \tag{2}$$

satisfies the equation (1).

If we consider the different version of $U_q(sl(2))$, the one with

$$u = (z - z^{-2})/(q - q^{-1}),$$

then, instead of (2), one should take

$$\mathfrak{v} = (q^2z^2 + z^{-2})(q^2 - 1)^{-1}(q - q^{-1})^{-1}. \tag{3}$$

C3.3. Example. Let now $A = k[z]$, $\vartheta f(z) = f(z+1)$. This subclass of algebras $A\langle\vartheta, u\rangle$ was introduced in [S] under the name *algebras similar to the enveloping algebra of* $sl(2)$. One can easily check that, if $\deg(u) \geq 1$, the equation (1) has unique solution $\mathfrak{v}$ such that $\mathfrak{v}(0) = 0$ (cf. [S], Lemma 1,4). ■

Let $R\{\theta, \xi\}$ be the associated with $A\langle\vartheta, u\rangle$ hyperbolic ring; i.e. $R = A[\xi]$, $\theta(\xi) = \xi + \vartheta(u)$, $\theta\mid_A = \vartheta$.

C3.4. Lemma. *Suppose* $A\langle\vartheta, u\rangle$ *is a ring of* $U(sl(2))$*-type with a weight* $\mathfrak{v}$ *. Then* $\xi - \mathfrak{v}$ *is a* θ*-invariant (hence central) element in* $R\{\theta, \xi\}$.

Proof. In fact,

$$\theta(\xi - \mathfrak{v}) = \xi + \vartheta(u) - \vartheta(\mathfrak{v}) = \xi + \vartheta(\mathfrak{v} - \vartheta^{-1}\mathfrak{v}) - \vartheta(\mathfrak{v}) = \xi - \mathfrak{v}. \quad ■$$

Fix a weight, $\mathfrak{v}$, of the ring $A\langle\vartheta, u\rangle$. And let η denote the θ-invariant element $\xi - \mathfrak{v}$. It is convenient to represent the ring $R = A[\xi]$ as $A[\eta]$ with $\theta\mid_A = \vartheta$ and $\theta(\eta) = \eta$, and with $\xi = \eta + \mathfrak{v}$.

Clearly, for every integer n, we have:

$$\theta^n(\xi) = \eta + \vartheta^n(\mathfrak{v}) = \xi + (\vartheta^n(\mathfrak{v}) - \mathfrak{v}) = \tag{4}$$

$$\theta^{-1}(\xi) + (\vartheta^n(\mathfrak{v}) - \vartheta^{-1}(\mathfrak{v}))$$

The elements $\vartheta^n(\mathfrak{v}) - \mathfrak{v}$ and $\vartheta^n(\mathfrak{v}) - \vartheta^{-1}(\mathfrak{v})$ do not depend on the choice of $\mathfrak{v}$, since

$$\vartheta^n(\mathfrak{v}) - \vartheta^{-1}(\mathfrak{v}) = \sum_{0\leq i\leq n} \vartheta^i(u) \tag{5}$$

and

$$\vartheta^{-n-1}(\mathfrak{v}) - \mathfrak{v} = \sum_{0\leq i\leq n} \vartheta^{-i}(u) \tag{6}$$

for every integer $n \geq 0$.

As we did in other cases, consider the corresponding to the element η decomposition of the left spectrum:

$$Spec_l R\{\theta, \xi\} = V_l(\eta) \bigcup U_l(\eta).$$

Since the element η is central, we have:

$$V_l(\eta) \simeq Spec_l A\{\vartheta, \mathfrak{v}\}; \quad U_l(\eta) \simeq Spec_l A[\eta, \eta^{-1}]\{\theta, \eta + \mathfrak{v}\}, \tag{7}$$

where the automorphism Θ is the trivial extension of ϑ, i.e. $\Theta\mid_A = \vartheta$, and $\Theta(\eta) = \eta$.

A straightforward application of results of Section 3 provides descriptions of both parts, $Spec_l A\{\vartheta, \mathfrak{v}\}$ and $Spec_l A[\eta, \eta^{-1}]\{\theta, \eta + \mathfrak{v}\}$, of the left spectrum of the ring $R\{\theta, \xi\}$. Details are left to a reader.

C4. Other examples of hyperbolic rings. There are lots of interesting hyperbolic rings which do not belong to any of the three classes discussed in Sections C1, C2, and C3. One of the best known of such rings is the *dispin algebra.*

C4.1. The dispin algebra. The *dispin* algebra, $U(osp(1,2))$, is the enveloping algebra of the Lie superalgebra $osp(1,2)$. It is generated by x, y, z satisfying the relations

$$zy - yz = y, \quad yx + xy = z, \quad xz - zx = x$$

Take $A = k[z]$, and define the automorphism ϑ by $\vartheta f(z) = f(z+1)$. Then algebra $U(osp(1,2)$ coincides with the algebra $A\langle\vartheta,\rho,u\rangle$, where $\rho = -1,\ u = z$. The corresponding hyperbolic algebra is $R\{\theta,\xi\}$, where

$$R = A[\xi] = k[z,\xi], \quad \theta\xi = -\xi + z + 1.$$

Clearly $R\{\theta,\xi\}$ cannot be of $M(2)$- or $U(sl(2))$-type

Since $\theta^{-1}(\xi) = -(\xi - z)$, we have:

$$\theta(\xi) + \theta^{-1}(\xi) = -\xi + z + 1 - \xi + z = (\rho + \rho^{-1})\xi + 2z + 1,$$

which shows that it is not of Heisenberg type either.

C4.2. Another deformation of $\mathbf{U(sl(2))}$. An example of a 'quantized' hyperbolic ring which is not of $M(2)$-, $U(sl(2))$-, or of Heisenberg type is the introduced by Woronowicz [W] deformation of $U(sl(2))$. This deformation, $\mathbb{W}(sl(2))$, is the k-algebra with generators x, y, z subject to the relations

$$xz - \nu^4 zx = (1+\nu^2)x,$$

$$xy - \nu^2 yx = \nu z,$$

$$zy - \nu^4 yz = (1+\nu^2)y,$$

where $\nu \in k^*$ is not a root of one.

We can rewrite these equations as

$$xz = (\nu^4 z + 1 + \nu^2)x$$

$$xy - \nu^2 yx = \nu z$$

$$zy = y(\nu^4 z + 1 + \nu^2).$$

Now it is clear that this Woronowicz's algebra is the algebra $A\langle\vartheta,\rho,u\rangle$, where

$$A = k[z], \quad \vartheta f(z) = f(\nu^4 z + 1 + \nu^2), \quad \rho = \nu^2, \quad u = \nu z.$$

The corresponding hyperbolic algebra, $R\{\theta,\xi\}$, is given by

$$R = k[z,\xi], \quad \theta\xi = \vartheta(\rho)\xi + \vartheta u = \nu^2\xi + \nu(\nu^4 z + 1 + \nu^2).$$

Since $\theta^{-1}\xi = \rho^{-1}(\xi - u) = \nu^{-2}(\xi - \nu z)$, we have:

$$\vartheta\xi + \theta^{-1}\xi = \nu^2\xi + \nu(\nu^4 z + 1 + \nu^2) + \nu^{-2}(\xi - \nu z) =$$

$$(\nu^2 + \nu^{-2})\xi + \nu((\nu^4 - \nu^{-2})z + 1).$$

Thus, the equality $(\vartheta + \vartheta^{-1})(\xi) = (\rho + \rho^{-1})\xi$ does not hold; i.e. $\mathbb{W}(sl(2))$ is not a ring of Heisenberg type.

C4.3. 3-Dimensional quasi-polynomial algebras. Both, the Woronowicz's algebra $\mathbb{W}(sl(2))$ and the dispin algebra $U(osp(1,2)$ are examples of algebras which were introduced in [BS] as *3-dimensional skew polynomial algebras.* To avoid confusion with the notion of a skew polynomial ring used in this chapter, we rename them into *quasi-polynomial* rings. By definition, a 3-dimensional quasi-polynomial k-algebra is given by the relations

$$yx - \alpha xy = \lambda, \quad zx - \beta xz = \mu, \quad xy - \gamma yx = \nu \tag{1}$$

such that

1) $\lambda, \mu, \nu \in kx + ky + kz + k$, and $\alpha, \beta, \gamma \in k^*$;

2) the 'standart monomials', $\{x^i y^j z^l \mid i, j, l \geq 0\}$, form a basis of the algebra.

C4.3.1. Theorem (2.5 in [BS]). *Let $\mathcal{A}$ be a 3-dimensional algebra defined by the relations* (1). *Up to isomorphism, $\mathcal{A}$ is given by the following relations:*

(a) if $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by

$$yz - \alpha zy = 0, \quad zx - \beta xz = 0, \quad xy - \gamma yx = 0$$

(b) if $|\{\alpha, \beta, \gamma\}| = 2$, and if $\beta \neq \alpha = \gamma = 1$, $\mathcal{A}$ is one of the following:

(**i**) $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$

(**ii**) $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$

(**iii**) $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$

(**iv**) $yz - zy = 0$, $zx - \beta xz = b$, $xy - yx = 0$

(**v**) $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$

(**vi**) $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$

Here $a, b \in k$ are arbitrary; all nonzero values of b yield isomorphic algebras.

(c) if $|\{\alpha, \beta, \gamma\}| = 2$, and if $\beta \neq \alpha = \gamma \neq 1$, then

(**i**) $yz - \alpha zy = 0$, $zx - \beta xz = y + b$, $xy - \alpha yx = 0$

(**ii**) $yz - \alpha zy = 0$, $zx - \beta xz = b$, $xy - \alpha yx = 0$

Here *$b \in k$ is arbitrary; all nonzero values of b yield isomorphic algebras.*

(d) if $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is given by

$$yz - \alpha zy = a_1 x + b_b$$

$$zx - \alpha xz = a_2 y + b_2$$

$$xy - \alpha yx = a_3 z + b_3$$

If $a_i = 0$, then all nonzero values of b_i yield isomorphic algebras.

(e) if $\alpha = \beta = \gamma = 1$, then $\mathcal{A}$ is isomorphic to one of the following algebras:

(i) $yz - zy = x$, $zx - xz = y$, $xy - yx = z$
(ii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = z$
(iii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = b$

iv) $yz - zy = -y$, $zx - xz = x + y$, $xy - yx = 0$
(v) $yz - zy = az$, $zx - xz = x$, $xy - yx = 0$

Here $a, b \in k$ *are arbitrary; all nonzero values of b yield isomorphic algebras.*

C4.4. 3-Dimensional skew polynomial and hyperbolic rings. The following algebras in the list of Theorem C4.3.1 are either hyperbolic or skew polynomial:

(b): algebras **(i)**, **(ii)** are hyperbolic, the algebra **(vi)** is skew polynomial;
(c): both algebras **(i)** and **(ii)** are hyperbolic;
(e): the algebra **(i)** is hyperbolic, the algebra **(v)** is skew polynomial**.**

We shall support this observation by producing the hyperbolic or skew polynomial structure for each of the listed above rings. Besides, we shall sketch the part of the left spectrum covered by Theorem 1.1.2 and Proposition 1.1.3 in the most interesting cases.

(b) **(i)** Take $A = k[y]$, $\vartheta f(y) = f(y-1)$. Then the relations **(i)** are

$$xa = \vartheta^{-1}(a)x, \quad za = \vartheta(a)z, \quad zx - \beta xz = y \in A;$$

i.e. $\mathcal{A} = A\langle \vartheta, \beta, y\rangle$. The corresponding hyperbolic ring is $R\{\theta, \xi\}$, where

$$R = k[y, \xi], \theta f(y, \xi) = f(y-1, \beta\xi + y - 1).$$

In particular, the dispin algebra (cf. C4.1) is isomorphic to the hyperbolic ring $R\{\theta, \xi\}$ for $\beta = -1$.

Clearly the ring $\mathcal{A}\langle \vartheta, 1, y\rangle$ (hence $R\{\theta, \xi\}$) is isomorphic to the enveloping algebra $U(sl(2))$. We assume that $\beta \neq 1$.

One can check that

$$\theta^n(\xi) = \beta^{n+1}\theta^{-1}(\xi) + \sum_{0 \le i \le n} \beta^i \vartheta^{n-i}(y) =$$

$$\beta^{n+1}\theta^{-1}(\xi) + \sum_{0 \le i \le n} \beta^i (y - n + i) = \tag{1}$$

$$\beta^{n+1}\theta^{-1}(\xi) + (1-\beta)^{-1}((1-\beta^{n+1})(y-n) + 1 - (n-1)\beta^n + (1-\beta^n)(1-\beta)^{-1})$$

and

$$\theta^{-n}(\xi) = \beta^{-n}\xi - \sum_{1 \le i \le n} \beta^{-i}\vartheta^{i-n}(y) =$$

$$\beta^{-n}\xi - \sum_{1\le i\le n}\beta^{-i}(y+n-i) = \beta^{-n}(\xi - \sum_{0\le i\le n-1}\beta^{i}(y+i)) = \tag{2}$$

$$\beta^{-n}\xi - \beta^{-n}(1-\beta)^{-1}((1-\beta^{n})y + \beta - n\beta^{n} + \beta(1-\beta^{n})(1-\beta)^{-1}).$$

Let the ideal $p \in SpecR$ contain $\theta^{-1}(\xi) = \beta^{-1}(\xi - y)$. It follows from (1) that $\theta^{n-1}(\xi) \in p$ if and only if

$$(1-\beta^{n})(y-n+1) + 1 - (n-2)\beta^{n-1} + (1-\beta^{n-1})(1-\beta)^{-1} \in p. \tag{3}$$

The formula (3) shows that, for every positive integer n such that $\beta^{n} \neq 1$, there exists unique prime ideal p such that

$$p_{1,n} := R\{\theta,\xi\}p + R\{\theta,\xi\}x + R\{\theta,\xi\}z^{n} \tag{4}$$

is a left ideal from $Spec_{l}R\{\theta,\xi\}$. This ideal p is

$$p(\lambda(n)) = R(\xi - \lambda(n)) + R(y - \lambda(n)), \tag{5}$$

where

$$\lambda(n) = n - 1 + (1-\beta^{n})^{-1}(1 - (n-2)\beta^{n-1} + (1-\beta^{n-1})(1-\beta)^{-1}).$$

Similarly, if $\beta^{n} \neq 1$, there exists unique prime ideal p such that

$$p_{n,1} := R\{\theta,\xi\}p + R\{\theta,\xi\}z + R\{\theta,\xi\}x^{n} \tag{6}$$

is a left ideal from $Spec_{l}R\{\theta,\xi\}$. This ideal p is

$$p(\lambda(-n)) = R\xi + R(y - \lambda(-n)), \tag{7}$$

where

$$\lambda(-n) = (1-\beta^{n})^{-1}(n\beta^{n} - \beta) - \beta(1-\beta)^{-1}.$$

Obviously, for every integer n, the ideal $p(\lambda(n))$ is maximal, and the quotient ring $R/p(\lambda(n))$ is isomorphic to k. Therefore the left ideal $p(\lambda(n))_{1,n}$ is maximal for every n, and the corresponding quotient module,

$$R\{\theta,\xi\}/p(\lambda(n))_{1,n},$$

has dimension n over k. And this exhausts the list of finite dimensional irreducible representations and even R-finite modules from the spectrum of the category $R\{\theta,\xi\} - mod$.

In particular, the list of equivalence classes of finite dimensional irreducible representations of the dispin algebra (the case when $\beta = -1$) contains one representation in every odd dimension and no even-dimensional representatons.

Every prime ideal p in $k[y]$ which does not contain $y + \lambda(n)$ (i.e.) is not equal to $k[y](y + \lambda a(n))$ for any n, defines two ideals from $Spec_{l}R\{\theta,\xi\}$:

$$R\{\theta,\xi\}(\xi - y) + R\{\theta,\xi\}\mathfrak{p} + R\{\theta,\xi\}x$$

and

$$R\{\theta,\xi\}\xi + R\{\theta,\xi\}\mathfrak{p} + R\{\theta,\xi\}z$$

Thus, the 'Verma' part of the left spectrum contains two non-closed points,

$$\mathbf{p}(0) = R\{\theta,\xi\}(\xi - y) + R\{\theta,\xi\}x \tag{8}$$

and

$$\mathbf{p}\hat{}(0) = R\{\theta,\xi\}\xi + R\{\theta,\xi\}z, \tag{9}$$

and two families of closed points (– maximal left ideals):

$$\mathbf{p}(f) := R\{\theta,\xi\}(\xi - y) + R\{\theta,\xi\}f + R\{\theta,\xi\}x \tag{10}$$

and

$$\mathbf{p}\hat{}(g) := R\{\theta,\xi\}\xi + R\{\theta,\xi\}g + R\{\theta,\xi\}z, \tag{11}$$

where $f = f(y)$ and $g = g(y)$ run through the set of all irreducible polynomials in y which are not equivalent to $y - \lambda(n)$ for any n.

In particular, if the field k is algebraically closed, then, instead of (8) and (9), we can write:

$$\mu(\lambda) := R\{\theta,\xi\}(\xi - \lambda) + R\{\theta,\xi\}(y - \lambda) + R\{\theta,\xi\}x \tag{12}$$

where λ runs through $k - \{\lambda(n) \mid n \geq 0\}$, and

$$\mu\hat{}(\lambda) := R\{\theta,\xi\}\xi + R\{\theta,\xi\}(x - \lambda) + R\{\theta,\xi\}z, \tag{13}$$

where λ runs through $k - \{\lambda(-n) \mid n \geq 1\}$.

Note that the family of ideals (10) is exactly the set of all nontrivial specializations of the ideal $\mathbf{p}(0)$ (cf. (8)), and the family of ideals (11) is the set of all nontrivial specializations of the ideal $\mathbf{p}\hat{}(0)$ (cf. (9)).

Now, suppose that p is a prime ideal in $R = k[y,\xi]$ such that $\theta^n p \neq p$ for every $n \neq 0$, and $\theta^n(\xi) \notin p$ for any n.

For example, the ideal $m(\gamma,\nu) := R(\xi - \gamma) + R(y - \nu)$, where

$$\gamma \neq 0, \gamma \neq (1 - \beta^{-n})(1 - \beta)^{-1}(1 + \lambda(n) - \nu), \tag{14}$$

$$\gamma \neq (1 - \beta^n)(1 - \beta)^{-1}(\nu - \lambda(-n)) \quad \text{for any } n \geq 1,$$

has this property.

Then the left ideal $p_{\infty,\infty} = R\{\theta,\xi\}p$ belongs to $Spec_l R\{\theta,\xi\}$. Moreover, if the ideal p is maximal, then $p_{\infty,\infty}$ is a maximal left ideal. In particular, the left ideals

$$m(\gamma,\nu)_{\infty,\infty} = R\{\theta,\xi\}(\xi - \gamma) + R\{\theta,\xi\}(y - \nu), \tag{15}$$

where the pair γ,ν satisfies the conditions (14), are maximal.

(b) **(ii)** $\mathcal{A} = A\{\vartheta, \beta, b\}$ for the same A and ϑ as in **(i)**, but with $b \in k^*$ instead of y. I.e. $\mathcal{A} \simeq R\{\theta, \xi\}$, where

$$R = k[y, \xi], \quad \theta f(y, \xi) = f(y-1, \beta\xi + b).$$

Thus, we have:

$$\theta^n(\xi) = \beta^{n+1}\theta^{-1}(\xi) + \sum_{0 \le i \le n} \beta^i \vartheta^{n-i}(b) =$$

$$\beta^{n+1}\theta^{-1}(\xi) + b(1-\beta)^{-1}(1-\beta^{n+1}),$$

$$\theta^{-n}(\xi) = \beta^{-n}\xi - \sum_{1 \le i \le n} \beta^{-i}\vartheta^{i-n}(y) =$$

$$\theta^{-n}(\xi) = \beta^{-n}(\xi - b(1-\beta)^{-1}(1-\beta^n)).$$

This time, the left ideal

$$p_{1,n} := R\{\theta, \xi\}p + R\{\theta, \xi\}x + R\{\theta, \xi\}z^n,$$

where $p \in SpecR$ and $\xi - y \in p$, belongs to $Spec_l R\{\theta, \xi\}$ if and only if $\beta^n = 1$, but $\beta^i \neq 1$ if $1 \le i \le n-1$.

Moreover, the left ideals

$$p_{1,\infty} := R\{\theta, \xi\}p + R\{\theta, \xi\}x \text{ and } | \text{ or } p_{\infty,1} := R\{\theta, \xi\}p + R\{\theta, \xi\}z$$

belong to $Spec_l R\{\theta, \xi\}$ if and only if β is not a root of one.

If $p \in SpecR$ is such that

$$\xi \pm b(1-\beta)^{-1}(1-\beta^n) \notin p$$

and $\theta^n(p) \neq p$ for any n, then $p_{\infty,\infty} = R\{\theta, \xi\}p$ belongs to the left spectrum. If p is a maximal ideal, then the left ideal $p_{\infty,\infty}$ is also maximal. In particular, the left ideal

$$R\{\theta, \xi\}(\xi - \gamma) + R\{\theta, \xi\}(y - \nu)$$

is maximal if $\gamma \neq \pm b(1-\beta)^{-1}(1-\beta^n)$ for any n.

(vi) Take $B = k[x, y]$, $\vartheta f(x, y) = f(\beta x, y+1)$. Then $\mathcal{A} = B[z; \vartheta]$.

(i) Take $A = k[y]$, $\vartheta f(y) = f(\alpha y)$, $u = y + b$. Then $\mathcal{A} = A\langle\vartheta, \beta, u\rangle$.

The corresponding hyperbolic ring is $R\{\theta, \xi\}$, where

$$R = k[y, \xi], \quad \theta f(y, \xi) = f(\alpha^{-1}y, \beta\xi + \alpha^{-1}y + b).$$

Note that Woronowicz's deformation of $U(sl(2))$ (cf. Example C4.2) belongs to this class: $\alpha = \nu^4$, $\beta = \nu^{-2}$, $b = \nu^2(\nu^2 - 1)^{-1}$.

We have:

$$\theta^n(\xi) = \beta^{n+1}\theta^{-1}(\xi) + \sum_{0\le i\le n} \beta^i \vartheta^{n-i}(y+b) =$$

$$\beta^{n+1}\theta^{-1}(\xi) + \sum_{0\le i\le n} \beta^i(\alpha^{i-n}y+b).$$

If $\alpha\beta = 1$, then it follows from the last expression that

$$\theta^{n-1}(\xi) = \beta^n\theta^{-1}(\xi) + n\beta^{n-1}y + b(1-\beta)^{-1}(1-\beta^n).$$

If $\alpha\beta \neq 1$, then

$$\theta^{n-1}(\xi) = \beta^n\theta^{-1}(\xi) + (1-\alpha\beta)^{-1}\alpha^{-n+1}(1-(\alpha\beta)^n)y + b(1-\beta)^{-1}(1-\beta^n).$$

Similarly,

$$\theta^{-n}(\xi) = \beta^{-n}\xi - \sum_{1\le i\le n} \beta^{-i}\vartheta^{i-n}(y) = \beta^{-n}(\xi - \sum_{0\le i\le n-1} \beta^i(\alpha^i y+b)),$$

which implies that

$$\theta^{-n}(\xi) = \beta^{-n}(\xi - ny - b(1-\beta)^{-1}(1-\beta^n))$$

if $\alpha\beta = 1$, and

$$\theta^{-n}(\xi) = \beta^{-n}(\xi - (1-\alpha\beta)^{-1}(1-(\alpha\beta)^n)y - b(1-\beta)^{-1}(1-\beta^n))$$

if $\alpha\beta \neq 1$.

For any integer $n \ge 1$, set

$$t(n) = n^{-1}\beta^{-n+1}b(1-\beta)^{-1}(1-\beta^n), t(-n) = n^{-1}b(1-\beta)^{-1}(1-\beta^n)$$

if $\alpha\beta = 1$, and

$$t(n) = (1-\alpha\beta)\alpha^{n-1}(1-(\alpha\beta)^n)^{-1}b(1-\beta)^{-1}(1-\beta^n),$$
$$t(-n) = (1-\alpha\beta)(1-(\alpha\beta)^n)^{-1}b(1-\beta)^{-1}(1-\beta^n)$$

if $(\alpha\beta)^n \neq 1$.

Let the ideal $p \in SpecR$ contain $\theta^{-1}(\xi) = \beta^{-1}(\xi - y - b)$. It follows from (16) that $\theta^{n-1}(\xi) \in p$ if and only if

either $\alpha^n = 1 = \beta^n$, or $(\alpha\beta)^n \neq 1$ and $y + t(n) \in p$.

(a) Let $\alpha\beta = 1$. And suppose that $\beta^n = 1$, but $\beta^i \neq 1$ if $1 \le i < n$. Then every $\mathbf{p} \in Spec_l R\{\theta,\xi\}$ such that the quotient module $R\{\theta,\xi\}/\mathbf{p}$ is R-finite, is equivalent to one of the left maximal ideals

$$\mathbf{p}_i := R\{\theta,\xi\}(\xi - t(i) - b) + R\{\theta,\xi\}(y + t(i)) + R\{\theta,\xi\}x + R\{\theta,\xi\}z^i$$

for every $1 \leq i \leq n$; and, if $i \neq j$, the ideals $\mathbf{p}_i, \mathbf{p}_j$ are not equivalent to each other.

If β is not a root of one, then, for every $i \geq 1$, the maximal left ideal $\mathbf{p}_i$ is in $Spec_l R\{\theta, \xi\}$; $\mathbf{p}_i$ is not equivalent to $\mathbf{p}_j$ if $i \neq j$, and every $\mathbf{p} \in Spec_l R\{\theta, \xi\}$ is equivalent to one of $\mathbf{p}_i$.

Clearly any prime ideal p in R which contains $\theta^{-1}(\xi)$ is of the form

$$p = R\theta^{-1}(\xi) + Rf = R(\xi - y - b) + Rf,$$

where $f = f(y)$ is an irreducible polynomial. The left ideal

$$p_{1,\infty} := R\{\theta, \xi\}(\xi - y - b) + R\{\theta, \xi\}f + R\{\theta, \xi\}x,$$

belongs to $Spec_l R\{\theta, \xi\}$ if and only if $f(y)$ is not equivalent to $y + t(i)$ for any i.

Similarly, the only ideals $p_{\infty,1}$ in $Spec_l R\{\theta, \xi\}$ (p is prime, and $\xi \in \theta(p)$) are of the form

$$R\{\theta, \xi\}\xi + R\{\theta, \xi\}f + R\{\theta, \xi\}z,$$

where $f = f(y)$ is an irreducible polynomial which is not equivalent to $y + t(-i)$ for any $i \geq 1$.

It is not difficult to describe the ideals $p \in SpecR$ such that $R\{\theta, \xi\}p \in Spec_l R\{\theta, \xi\}$ (cf. Proposition 3.2.3):

Every $p \in SpecR$ such that $\theta^n(\xi) \notin p$ belongs to this set except
the ideals Rg for any irreducible $g \in k[y]$, if β is a root of one;
the ideal Ry, if β is not a root of one.

(b) Let now $\alpha\beta \neq 1$. And suppose that $\alpha^n = 1 = \beta^n$, but the condition $\alpha^i = 1 = \beta^i$ does not hold for $1 \leq i \leq n - 1$. Then

$$\mathfrak{p}(0)_{1,n} := R\{\theta, \xi\}(\xi - y - b) + R\{\theta, \xi\}x + R\{\theta, \xi\}z^n$$

belongs to $Spec_l R\{\theta, \xi\}$. The set of nontrivial specializations of the ideal $\mathfrak{p}(0)_{1,n}$ consists of all the ideals

$$\mathfrak{p}(f)_{1,n} := R\{\theta, \xi\}(\xi - y - b) + R\{\theta, \xi\}f + R\{\theta, \xi\}x + R\{\theta, \xi\}z^n,$$

where $f = f(y)$ is any irreducible polynomial which is not equivalent to $y + t(i)$ for some $1 \leq i < n$ such that $(\alpha\beta)^i \neq 1$.

Clearly all the left ideals $\mathfrak{p}(f)_{1,n}$, $f \neq 0$, are maximal.

Besides, there are maximal left ideals

$$\mathfrak{p}_{1,i} := R\{\theta, \xi\}(\xi - t(i) - b) + R\{\theta, \xi\}(y + t(i)) + R\{\theta, \xi\}x + R\{\theta, \xi\}z^i,$$

for every i such that $1 \leq i < n$ and $(\alpha\beta)^i \neq 1$.

The ideals $\mathfrak{p}(0)_{1,n}$, $\{\mathfrak{p}(f)_{1,n}\}$ and $\{\mathfrak{p}_{1,i}\}$ are not equivalent one to another, and every ideal $\mathbf{p} \in Spec_l R\{\theta, \xi\}$ such that the module $R\{\theta, \xi\}/\mathbf{p}$ is R-finite, is equivalent to one of them.

Since $\theta^n = id$, the series $\{p_{1,\infty}\}$, $\{p_{\infty,1}\}$, and $\{p_{\infty,\infty}\}$ are, evidently, empty.

(c) Now assume that $\alpha\beta \neq 1$, and the condition $\alpha^n = 1 = \beta^n$ does not hold for any $n \geq 1$. Then, for every i such that $(\alpha\beta)^i \neq 1$, the left ideal

$$\mathfrak{p}_{1,i} := R\{\theta,\xi\}(\xi - t(i) - b) + R\{\theta,\xi\}(y + t(i)) + R\{\theta,\xi\}x + R\{\theta,\xi\}z^i,$$

is maximal. Every $\mathbf{p} \in Spec_l R\{\theta,\xi\}$ such that the quotient module $R\{\theta,\xi\}/\mathbf{p}$ is R-finite, is equivalent to one of the ideals $\mathfrak{p}_{1,i}$.

The series $\{p_{1,\infty}\}$ and $\{p_{\infty,1}\}$ consist of the ideals

$$R\{\theta,\xi\}(\xi - y - b) + R\{\theta,\xi\}f + R\{\theta,\xi\}x,$$

$$R\{\theta,\xi\}\xi + R\{\theta,\xi\}g + R\{\theta,\xi\}z,$$

where f (resp. g) runs through the set of all irreducible polynomials in y which are not equivalent to $y + t(i)$ (resp. to $y + t(-i)$) for any $i \geq 1$.

Suppose that $p \in SpecR$ is such that $\theta^n(\xi) \notin p$ for any n. Then $p_{\infty,\infty} = R\{\theta,\xi\}p \in Spec_l R\{\theta,\xi\}$ provided

$p \neq Ry$, if α is not a root of one;

$p \neq Rg$ for any irreducible polynomial $g(y)$, if α is a root of one.

Moreover, if the ideal p is maximal, then $p_{\infty,\infty}$ is a maximal left ideal. In particular, the left ideals

$$m(\gamma,\nu)_{\infty,\infty} = R\{\theta,\xi\}(\xi - \gamma) + R\{\theta,\xi\}(y - \nu),$$

where $\gamma \neq t(i)$ for any integer i, are maximal.

(ii) $\mathcal{A} = A\langle\vartheta,\beta,b\rangle$ where $A = k[y]$, $\vartheta f(y) = f(\alpha y)$.

The corresponding hyperbolic ring is $R\{\theta,\xi\}$, where

$$R = k[y,\xi], \quad \theta f(y,\xi) = f(\alpha^{-1}y, \beta\xi + b).$$

Here the formulas for $\theta^{\pm n}(\xi)$ are the same as in the case *(b)* **(ii)** :

$$\theta^n(\xi) = \beta^{n+1}\theta^{-1}(\xi) + b(1-\beta)^{-1}(1-\beta^{n+1}),$$

$$\theta^{-n}(\xi) = \theta^{-n}(\xi) = \beta^{-n}(\xi - b(1-\beta)^{-1}(1-\beta^n)).$$

We leave the description of $Spec_l R\{\theta,\xi\}$ to the reader as an exercise.

(e) **(i)** This is the enveloping algebra of the Lie algebra with basis x, y, z and the relations

$$[y,z] = x, \ [z,x] = y, \ [x,y] = z.$$

In other words, $\mathcal{A} = U(sl(2))$.

(e) **(ii)** This is the universal enveloping algebra of the Heisenberg Lie algebra. So, it can be regarded either as the hyperbolic ring $R\{\theta,\xi\}$, where $R = k[z,\xi]$ and $\theta f(z,\xi) = f(z,\xi + z)$ for any $f \in R$.

(v) Let $A = k[x,y]$, $\vartheta f(x,y) = f(x+1, y-a)$. Obviously, $\mathcal{A} = A[z;\vartheta]$.

C4.5. 3-Dimensional rings of skew differential operators. Let ϑ be an automorphism of a ring A; and let ∂ be a ϑ*-derivative*; i.e. ∂ is an additive map from A to A such that

$$\partial(ab) = \partial(a)b + \vartheta(a)\partial(b)$$

for all a, b in A. Recall that an Ore extension of a ring A defined by ϑ and ∂ is the ring $A[x, \vartheta, \partial]$ generated by A and the indeterminable x subject to the relations:

$$xa = \vartheta(a)x + \partial(a) \quad \textit{for all } a \in A \tag{1}$$

Clearly $A[x, \vartheta, 0]$ coincides with the skew polynomial ring $A[x, \vartheta]$.

Generic Ore extensions are called sometimes *skew polynomial rings.* However, the difference between geometrical pictures (the left spectrum, simple modules etc.) in the degenerate case, $\partial = 0$, and non-degenerate case turns out to be considerable enough to split these two cases. So, the ring of *skew differential operators (with coefficients in A)* seems to be more adequate version of a second name for Ore extensions.

The left spectrum and irreducible representations of rings of skew differential operators are pretty well understood in the case when the ring of coefficients is commutative and noetherian (cf. [R4], [R5]).

Now, resuming the contemplation of the list of algebras in Theorem C4.3.1, note that five of them - *(b)* **(iii)** and **(iv)**, and *(e)* **(ii)**, **(iii)**, **(iv)** - are rings of skew differential operators:

(b) **(iii)** Take $B = k[x, y]$, $\vartheta f(x, y) = f(\beta x, y)$, and define a ϑ-derivation δ by $\delta(x) = y, \delta(y) = 0$. Then $\mathcal{A} = B[z; \vartheta, \delta]$.

(b) **(iv)** $\mathcal{A} = B[z; \vartheta, \partial]$, where B and ϑ are as in **(iii)** and the ϑ-derivation ∂ is defined by $\partial(x) = b$, $\partial(y) = 0$.

(e) **(iii)** $\mathcal{A} = A[x; id, \delta]$, where $A = k[y, z]$, as in **(ii)**, and $\delta(y) = b$, $\delta(z) = 0$.

(e) **(iv)** Take $A = k[x, y]$, $\partial(y) = y$, $\partial(x) = x + y$. Then $\mathcal{A} = A[z; id, \partial]$.

One can apply to these algebras the results of [R4] and obtain a description of the left spectrum and irreducible representations.

C4.6. The remaining cases. The only algebras left from the list of Theorem C4.3.1 are: the 'generic' 3-dimensional algebra *(a)*, the algebra *(b)* **(v)** and, finally, the algebras *(d)*.

(a) Let $A = k[z]$, $\vartheta f(z) = f(\alpha z)$ for every $f \in A$; and let Θ be the automorphism of the algebra $B = A[y, \vartheta]$ which assigns to a polynomial $g(y, z)$ the polynomial $g(\gamma y, \beta^{-1} z)$. It is easy to see that $\mathcal{A} = B[x, \Theta]$.

(b) **(v)** Let $A = k[y]$, $\vartheta f(y) = f(y+1)$; and let θ be the automorphism of $A' = A[x; \vartheta]$ defined by $\theta g(x, y) = g(\beta x, y + a)$. Clearly $\mathcal{A} = A'[z; \theta]$.

Thus, in both cases, *(a)* and *(b)* **(v)**, the ring $\mathcal{A}$ is a double skew polynomial extension of a commutative (polynomial) ring.

The invariant (categorical) approach to the noncommutative algebraic geometry (cf. Chapter IV) allows to describe the left spectrum of iterated skew polynomial extensions.

(d) Suppose now that the algebra $\mathcal{A}$ belongs to the class *(d)*; i.e. it is defined by the relations

$$yz - \alpha zy = a_1 x + b_1 := \lambda$$

$$zx - \alpha xz = a_2 y + b_2 := \mu$$

$$xy - \alpha yx = a_3 z + b_3 := \nu$$

where $\alpha = \beta = \gamma \neq 1$.

C4.6.1. A special case. Let $a_3 = 0$. Define the automorphism ϑ of the algebra $A = k[y]$ by $\vartheta f(y) = f(\alpha y)$, and the ϑ-derivative ∂ by $\partial(y) = b_3$.

Now, define the automorphism Θ and the Θ-derivative δ of the ring $B = A[x, \vartheta, \partial]$ by

$$\Theta g(x, y) = g(\alpha x, \alpha^{-1} y), \quad \delta(x) = a_2 y + b_2, \delta(y) = -\alpha^{-1}(a_1 x + b_1).$$

Clearly our ring coincides with the Ore extension $B[z, \Theta, \delta]$.

In other words, if one of the coefficients a_i is zero, then the ring is a double Ore extension. Again, there is a way to get a pretty ample information about the left spectrum of a double Ore extension.

Thus the only case which remains, apparently, out of reach of the presented in this chapter technique (as well as [BS]) is when all the coefficients a_i are nonzero.